

Ruelle Operator for Continuous Potentials and DLR-Gibbs Measures

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Abstract

In this work we study the Ruelle Operator associated to continuous potentials on general compact spaces. We present a new criterion for uniqueness of the eigenmeasures associated to the spectral radius of this operator acting on the space of all real-valued continuous functions defined on some suitable compact metric space. This uniqueness result is obtained by showing that set of eigenmeasures for any continuous potential, coincides with the set of the DLR-Gibbs measures for a suitable quasilocal specification. In particular, we proved that the phase transition in the DLR sense is equivalent to existence of more than one eigenprobability for the dual of the Ruelle operator. The strongly non-null condition, for the specifications considered here, is analyzed, in the case of finite alphabets and on the Thermodynamic Formalism setting.

We also consider bounded extensions of the Ruelle operator to the Lebesgue space of integrable functions with respect to the eigenmeasures. We give very general sufficient conditions for the existence of integrable and almost everywhere positive eigenfunctions associated to the spectral radius of the Ruelle operator. A perturbation theory is developed for sequences of Ruelle operators and some continuity results are obtained, among them a limit theorem for the topological pressure functional. Techniques of super and sub solutions to the eigenvalue problem is developed and employed to construct (semi-explicitly) eigenfunctions for a very large class of continuous potentials having low regularity.

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1 Introduction

The classical Ruelle operator needs no introduction and nowadays is a key concept in the Thermodynamic Formalism. It was first defined in infinite dimensions in 1968 by David Ruelle in a seminal paper [20] and since then has attracted the attention of the Dynamical System community. Remarkable applications of this operator to Hyperbolic dynamical systems and Statistical Mechanics were presented by Ruelle, Sinai and Bowen, see [6, 20, 22]. The Ruelle operator was generalized in several directions and its generalizations are commonly called transfer operators. Nowadays transfer operators are present in several applications in pure and applied mathematics and is a fruitful area of active development, see [2] for comprehensive overview of the works before the two thousands.

The classical theory of Ruelle operator with the dynamics given by the full shift $\sigma : \Omega \rightarrow \Omega$ is formulated in the symbolic space $\Omega \equiv M^{\mathbb{N}}$, where $M = \{1, \dots, n\}$ with the operator acting on the space of all real-valued γ -Hölder continuous functions defined on Ω , which is denoted here by $C^\gamma(\Omega)$. On its classical form, given a continuous function $f : \Omega \rightarrow \mathbb{R}$, the Ruelle operator \mathcal{L}_f is such that $\mathcal{L}_f(\varphi) = \psi$, where the function ψ for any $x \in \Omega$ is given by

$$\psi(x) = \sum_{y \in \Omega: \sigma(y)=x} e^{f(y)} \varphi(y).$$

A central problem in this setting is to determine the solutions of the following variational problem proposed by Ruelle 1967, [19] and Walters 1975, [24]:

$$\sup_{\mu \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \{h(\mu) - \int_\Omega f d\mu\}, \quad (1)$$

where $h(\mu)$ is the Kolmogorov-Sinai entropy of μ and $\mathcal{P}_\sigma(\Omega, \mathcal{F})$ is the set of all σ -invariant Borel probability measures over Ω and \mathcal{F} is the sigma algebra generated by the cylinder sets.

A probability which attains this supremum value is called an equilibrium state for the potential $-f$. It is well known that the Ruelle operator \mathcal{L}_{-f} is quite useful for getting equilibrium states (see [2] and [18]). We remark that the simple existence of the solution to the variational problem can be easily obtained through abstract theorems of convex analysis, but as we said the Ruelle operator approach to this problem gives us much more information about the solution as uniqueness, for instance. One can also get differentiability properties with variation of the potential using this approach (see [4]). But that comes with a cost which is the determination of certain spectral properties of the Ruelle operator. For the symbolic space we mentioned above all these problems were overcome around the eighties and nineties and in that time much more general potentials were considered, see [5, 25, 26, 27] for their precise definitions. These new spaces of potentials are denoted by $W(\Omega, \sigma)$ and $B(\Omega, \sigma)$, where σ in the notation refers to the dynamics and is particularized here to the left shift map. Potentials on $W(\Omega, \sigma)$ are said to satisfy the Walters condition and those in $B(\Omega, \sigma)$ Bowen's condition. We shall remark that $C^\gamma(\Omega) \subset W(\Omega, \sigma) \subset B(\Omega, \sigma)$, but the spectral analysis of the Ruelle operator is more efficient in $C^\gamma(\Omega)$ because of the spectral gap property of the Ruelle

operator acting in this space. Although this operator has no spectral gap in $W(\Omega, \sigma)$ and $B(\Omega, \sigma)$ the Ruelle operator approach is still useful to attack the above variational problem and provides rich connections among symbolic dynamics, Statistical Mechanics and probability theory.

The above mentioned works of Peter Walters are not restricted to dynamics given by the shift mapping and spaces like infinite cartesian products of finite sets. Basically he considered expanding mappings $T : \Omega \rightarrow \Omega$ on compact spaces satisfying the property that the number of preimages of each point is finite. So symbolic spaces like $\Omega = (\mathbb{S}^1)^\mathbb{N}$ with the shift acting on it does not fit his theory. The problem is related to the fact that the number of preimages under the shift map is not countable. Let us recall that several famous models of Statistical Mechanics are defined over the alphabet \mathbb{S}^{n-1} , the unit sphere of \mathbb{R}^n (which is uncountable for $n \geq 2$). For example, $n = 0$ give us the Self-Avoiding Walking (SAW), $n = 1$ is the Ising model, $n = 2$ (the first uncountable example on this list) the so-called XY model, for $n = 3$ we have the Heisenberg model and for $n = 4$ the toy model for the Higgs sector of the Standard Model, see [1, 3, 13, 14, 15, 16, 23] for more details.

In [3] the authors used the idea of an *a priori* measure $p : \mathcal{B}(\mathbb{S}^1) \rightarrow [0, 1]$ to circumvent the problem of uncountable alphabets and developed the theory of the Ruelle operator for Hölder potentials on $(\mathbb{S}^1)^\mathbb{N}$ with the dynamics given by the left shift map. The scheme developed to handle the $(\mathbb{S}^1)^\mathbb{N}$ case, works similarly for Hölder potentials when one replaces the unit circle \mathbb{S}^1 by a more general compact metric space M , but in the general case we have to be careful about the choice of the *a priori* measure $p : \mathcal{B}(M) \rightarrow [0, 1]$, see [17] for details. In this more general setting the operator is defined as

$$\mathcal{L}_f(\varphi)(x) = \int_M e^{f(ax)} \varphi(ax) dp(a),$$

where $ax := (a, x_1, x_2, \dots)$. A full support condition is imposed on the *a priori* measure in [17] but, this is not a strong restriction, since in the majority of the applications there is a natural choice for the *a priori* measure and it always satisfies this full support condition. For instance, in the classical Ruelle operator \mathcal{L}_f the metric space is some finite set as $M = \{1, 2, \dots, n\}$ and one normally consider the normalized counting measure as the *a priori* measure p on M . When M is a general compact group one can consider the normalized Haar Measure. For example, if $M = \mathbb{S}^1$, then one can consider the Lebesgue measure dx on \mathbb{S}^1 as a natural choice for the *a priori* probability measure, see [3].

Another progress towards considering more general potentials defined, on infinite cartesian products of a general metric compact spaces, was obtained by one of the authors in [10]. In this work a version of the Ruelle-Perron-Fröbenius theorem is obtained for what the authors called weak and strong Walters conditions, which are natural generalizations of the classical Walters condition.

Some results mentioned above has its counterpart in case M is a countable infinite set but not compact when regarded as topological space. The Thermodynamic Formalism for such alphabets are motivated in parts by application to non-uniformly hyperbolic dynamical systems [21] and references therein.

When $M = \mathbb{S}^1$, for example, the left shift mapping losses the important dynamical property which is the uniformly expandibility. This is intimately connected to the existence and uniformly convergence of

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_f^n(1)(x)$$

to the topological pressure and also in studying differentiability and analyticity of the Thermodynamic quantities. Some progress regarding these problems are obtained, in [10].

The goal of the present paper is to study the Ruelle operator \mathcal{L}_f associated to a general continuous potential f defined over an infinite cartesian product of a general compact metric space. One of the main results here is the extension of the results mentioned above to potentials satisfying a Bowen's like condition (see Theorem 5).

We also construct a Perturbation theory for the Ruelle operator in the sense of $C(\Omega)$ perturbations and present a constructive approach to solve the classical variational problem, for continuous potentials.

In Statistical Mechanics (and also Thermodynamic Formalism) the problem of existence and multiplicity of DLR Gibbs Measures play a very important role (see [21]). An analysis regarding the uniqueness in the Thermodynamic Formalism setting was done in [8], in case where the state space is finite for a large class of continuous potentials. The study of the multiplicity of the DLR Gibbs Measures is an important problem in Statistical Mechanics and Thermodynamic Formalism because of the Dobrushin interpretation of it as phase transitions. There is no universal definition of what a phase transition is but nowadays it is understood as either the existence of more than one DLR state, more than one eigenprobability for the dual of the Ruelle operator or, more than one equilibrium state and so on (see [9, 12] for more details). These concepts and their connections will be carefully described here and this is helpful for understanding when there exist or not phase transitions.

This paper is structured as follows. In section 2 the Ruelle Operator acting on $C(\Omega)$, where Ω is a infinite cartesian product of an arbitrary compact metric space, is introduced. We recall the standard strategy used to obtain the maximal eigenvalue λ_f (the maximality is discussed in further section) of \mathcal{L}_f as well as the probability measures ν_f satisfying $\mathcal{L}_f^*(\nu_f) = \lambda_f \nu_f$.

In section 3 we obtain an intrinsic formulae for the asymptotic pressure for arbitrary continuous potential. This result is similar to the one obtained by Peter Walters in [26] but use different approach since the entropy argument used there can not be directly applied for the case of general compact metric space M as we are considering here. In this section we also prove that the classical strategy above mentioned to construct ν_f always provide us eigenprobabilities associated to the maximal eigenvalue λ_f .

Section 4 establish in full generality the equivalence between eigenmeasures and DLR-Gibbs measures for any continuous potential. In particular, we explain how to construct a quasilocal specification from the continuous potential f and the Ruelle operator. So letting clear what kind of results one can immediately import from the DLR-Gibbs Measures to the Thermodynamic Formalism.

In sections 5 and 6 we prove the quasilocal property of the specifications introduced in this work, and the strongly non-null condition is analyzed when the state space M is finite, respectively. In Section 6 is presented an example where the specifications constructed here from a continuous potential is not strongly non-null. This example in finite state space is rather surprising.

At the end, in section 7 we obtain a kind of Bowen criteria for the uniqueness of the eigenprobabilities applicable to a general compact metric state space. This theorem is non trivial generalization of the classical ones (Hölder, Walters and Bowen).

Section 8 the extension of \mathcal{L}_f to the Lebesgue space $L^1(\nu_f)$ is considered. We show that the operator norm is given by λ_f thus proving the maximality of the eigenvalue λ_f obtained in the Section 2. We also point out that the classical duality relation $\int_{\Omega} \mathcal{L}_f(\varphi) d\nu_f = \lambda_f \int_{\Omega} \varphi d\nu_f$ extends naturally for test functions $\varphi \in L^1(\nu_f)$.

In Section 9 we prove continuity results for sequences of Ruelle operators in the uniform operator norm.

In Section 10 we give very general conditions for the existence of the eigenfunctions in the L^1 sense for potentials having less regularity than Hölder, Walters and Bowen, for example. Under mild hypothesis on the potential we prove that $\limsup h_{f_n}$ (here f_n is a sequence of potentials converging uniformly to f) is a non trivial Lebesgue integrable (with respect to any eigenprobability) eigenfunction associated to the maximal eigenvalue. Another remarkable result concerning to the eigenfunctions is the proof that $\limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)/\lambda_f^n$ is non trivial eigenfunction of \mathcal{L}_f under fairly general conditions.

In Section 11 we present applications of our previous results. The first application concerns to cluster points of the sequence of eigenprobabilities $(\mu_{f_n})_{n \in \mathbb{N}}$, where f_n is a suitable truncation of f . We prove that such cluster points belongs to the set of eigenprobabilities for f . In this application f_n and f are assumed to have Hölder regularity.

2 Preliminaries

Here and subsequently (M, d) denotes a compact metric space endowed with a Borel probability measure μ which is assumed to be fully supported in M . Let Ω denote the infinite cartesian product $M^{\mathbb{N}}$ and \mathcal{F} be the σ -algebra generated by its cylinder sets. We will consider the dynamics on Ω given by the left shift map $\sigma : \Omega \rightarrow \Omega$ which is defined, as usual, by $\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots)$. We use the notation $C(\Omega)$ for the space of all real continuous functions on Ω . When convenient we call an element $f \in C(\Omega)$ a potential and unless stated otherwise all the potentials are assumed to be a general continuous function. The Ruelle operator associated to the potential f is a mapping $\mathcal{L}_f : C(\Omega) \rightarrow C(\Omega)$ that sends $\varphi \mapsto \mathcal{L}_f(\varphi)$ which is defined for each $x \in \Omega$ by the following expression

$$\mathcal{L}_f(\varphi)(x) = \int_M \exp(f(ax)) \varphi(ax) d\mu(a), \quad \text{where } ax := (a, x_1, x_2, \dots). \quad (2)$$

Due to compactness of Ω , in the product topology and the Riesz-Markov theorem we have that $C^*(\Omega)$ is isomorphic to $\mathcal{M}_s(\Omega, \mathcal{F})$, the space of all signed Radon measures.

Therefore we can define \mathcal{L}_f^* , the dual of the Ruelle operator, as the unique continuous map from $\mathcal{M}_s(\Omega, \mathcal{F})$ to itself satisfying for each $\gamma \in \mathcal{M}_s(\Omega, \mathcal{F})$ the following identity

$$\int_{\Omega} \mathcal{L}_f(\varphi) d\gamma = \int_{\Omega} \varphi d(\mathcal{L}_f^* \gamma) \quad \forall \varphi \in C(\Omega). \quad (3)$$

From the positivity of \mathcal{L}_f follows that the map $\gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1)$ sends the space of all Borel probability measures $\mathcal{P}(\Omega, \mathcal{F})$ to itself. Since $\mathcal{P}(\Omega, \mathcal{F})$ is convex set and compact in the weak topology (which is Hausdorff in this case) and the mapping $\gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1)$ is continuous the Schauder-Tychonoff theorem ensures the existence of at least one Borel probability measure ν such that $\mathcal{L}_f^*(\nu) = \mathcal{L}_f^*(\nu)(1) \cdot \nu$. Notice that this eigenvalue $\lambda \equiv \mathcal{L}_f^*(\nu)(1)$ is positive but strictly speaking it could depend on the choice of the fixed point when it is not unique, however any case such eigenvalues trivially satisfies $\exp(-\|f\|_{\infty}) \leq \lambda \leq \exp(\|f\|_{\infty})$ so we can always work with

$$\lambda_f = \sup \left\{ \mathcal{L}_f^*(\nu)(1) : \begin{array}{l} \nu \in \mathcal{P}(\Omega, \mathcal{F}) \text{ and } \nu \text{ is fix point for} \\ \gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1) \end{array} \right\}. \quad (4)$$

Of course, from the compactness of $\mathcal{P}(\Omega, \mathcal{F})$ and continuity of \mathcal{L}_f^* the supremum is attained and therefore the set defined below is not empty.

Definition 1 ($\mathcal{G}^*(f)$). *Let f be a continuous potential and λ_f given by (4). We define*

$$\mathcal{G}^*(f) = \{\nu \in \mathcal{P}(\Omega, \mathcal{F}) : \mathcal{L}_f^* \nu = \lambda_f \nu\}.$$

To study the eigenfunctions of \mathcal{L}_f , where f is a general continuous potential, we will need the RPF theorem for the Hölder class. This theorem is stated as follows, see [3] and [17] for the proof.

We consider the metric d_{Ω} on Ω given by $d_{\Omega}(x, y) = \sum_{n=1}^{\infty} 2^{-n} d(x_n, y_n)$ and for any fixed $0 < \gamma \leq 1$ we denote by $C^{\gamma}(\Omega)$ the space of all γ -Hölder continuous functions, i.e, the set of all functions $\varphi : \Omega \rightarrow \mathbb{R}$ satisfying

$$\text{Hol}_{\gamma}(\varphi) = \sup_{x, y \in \Omega : x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d_{\Omega}(x, y)^{\gamma}} < +\infty.$$

Theorem 1 (Ruelle-Perron-Fröbenius: Hölder potentials). *Let (M, d) be a compact metric space, μ a Borel probability measure of full support on M and f be a potential in $C^{\gamma}(\Omega)$, where $0 < \gamma < 1$. Then $\mathcal{L}_f : C^{\gamma}(\Omega) \rightarrow C^{\gamma}(\Omega)$ have a simple positive eigenvalue of maximal modulus λ_f and there are a strictly positive function h_f satisfying $\mathcal{L}_f(h_f) = \lambda_f h_f$ and a Borel probability measure ν_f for which $\mathcal{L}_f^*(\nu_f) = \lambda_f \nu_f$ and $\mathcal{L}_f^*(\nu_f)(1) = \lambda_f$.*

3 The Pressure of Continuous Potentials

Proposition 1. *Let $f \in C(\Omega)$ be a potential and λ_f given by (4). Then, for any $x \in \Omega$ we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_f^n(1)(\sigma^n x) = \log \lambda_f.$$

Proof. Let $\nu \in \mathcal{G}^*(f)$ a fixed eigenprobability. Without loss of generality we can assume that $\text{diam}(M) = 1$. By the definition of d_Ω for any pair $z, w \in \Omega$ such that $z_i = w_i$, $\forall i = 1, \dots, N$ we have that $d_\Omega(z, w) \leq 2^{-N}$. From uniform continuity of f given $\varepsilon > 0$, there is $N_0 \in \mathbb{N}$, so that $|f(z) - f(w)| < \varepsilon/2$, whenever $d_\Omega(z, w) < 2^{-N_0}$. If $n > 2N_0$ and $a := (a_1, \dots, a_n)$ we claim that

$$\|S_n(f)(ax) - S_n(f)(ay)\|_\infty \leq (n - N_0)\frac{\varepsilon}{2} + 2\|f\|_\infty N_0. \quad (5)$$

Indeed, for any $n \geq 2N_0$ we have

$$\begin{aligned} |S_n(f)(ax) - S_n(f)(ay)| &= \left| \sum_{j=1}^n f(\sigma^j(a_1, \dots, a_n, x)) - \sum_{j=1}^n f(\sigma^j(a_1, \dots, a_n, y)) \right| \\ &\leq \sum_{j=1}^{n-N_0} |f(\sigma^j(a_1, \dots, a_n, x)) - f(\sigma^j(a_1, \dots, a_n, y))| + \\ &\quad \sum_{j=1}^{N_0} |f(\sigma^j(a_{n-N_0}, \dots, a_n, x)) - f(\sigma^j(a_{n-N_0}, \dots, a_n, y))| \\ &\leq (n - N_0)\frac{\varepsilon}{2} + 2N_0\|f\|_\infty. \end{aligned}$$

The last inequality comes from the uniform continuity for the first terms and from the uniform norm of f for the second ones.

We recall that for any probability space (E, \mathcal{E}, P) , φ and ψ bounded real \mathcal{E} -measurable functions the following inequality holds

$$\left| \log \int_E e^{\varphi(\omega)} dP(\omega) - \log \int_E e^{\psi(\omega)} dP(\omega) \right| \leq \|\varphi - \psi\|_\infty. \quad (6)$$

From the definition of the Ruelle operator, for any $n \in \mathbb{N}$, we have

$$\mathcal{L}_f^n(1)(\sigma^n x) = \int_{M^n} \exp(S_n(f)(a\sigma^n x)) \prod_{i=1}^n d\mu(a_i)$$

and from (5) and (6) with $\varphi(a) = \exp(S_n(f)(a\sigma^n x))$ and $\psi(a) = \exp(S_n(f)(ay))$ we get for $n \geq \max\{N_0, 4\varepsilon^{-2}\|f\|_\infty^2\}$ the following inequality

$$\frac{1}{n} |\log(\mathcal{L}_f^n(1)(\sigma^n x)) - \log(\mathcal{L}_f^n(1)(y))| \leq \frac{1}{n} ((n - N_0)\frac{\varepsilon}{2} + \frac{2\|f\|_\infty N_0}{n}) \leq \varepsilon.$$

By using Fubini's theorem, sum and subtract $\exp(S_n(f)(ay))$, the identity (3) iteratively and the last inequality for $n \geq \max\{N_0, 4\varepsilon^{-2}\|f\|_\infty^2\}$ we obtain

$$\mathcal{L}_f^n(1)(\sigma^n x) = \int_{M^n} \exp(S_n(f)(a\sigma^n x)) \prod_{i=1}^n dp(a_i)$$

$$\begin{aligned}
&= \int_{M^n} \int_{\Omega} \exp(S_n(f)(a\sigma^n x)) d\nu(y) \prod_{i=1}^n dp(a_i) \\
&\leq \exp((n - \sqrt{n})\frac{\varepsilon}{2} + 2\|f\|_{\infty}\sqrt{n}) \int_{M^n} \int_{\Omega} \exp(S_n(f)(ay)) d\nu_f(y) \prod_{i=1}^n dp(a_i) \\
&\leq \exp(n\varepsilon) \int_{\Omega} \mathcal{L}_f^n(1)(y) d\nu(y) \\
&= \exp(n\varepsilon) \lambda_f^n.
\end{aligned}$$

Similarly we obtain the lower bound $\mathcal{L}_f^n(1)(\sigma^n x) \geq \exp(-n\varepsilon) \lambda_f^n$ so the proposition follows. \square

Corollary 1. *Let f be a continuous potential. If ν and $\hat{\nu}$ are fixed points for the map $\gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1)$ then $\mathcal{L}_f^*(\nu)(1) = \mathcal{L}_f^*(\hat{\nu})(1) = \lambda_f$.*

Proof. For any $x_0 \in \Omega$ by repeating the same steps of the proof of the above previous proposition one shows that $\log(\mathcal{L}_f^*(\nu)(1)) \equiv \log(\lambda_f(\nu)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_f^n(1)(x_0) = \log(\lambda_f(\hat{\nu})) = \log(\mathcal{L}_f^*(\hat{\nu})(1))$. \square

Definition 2 (The Pressure Functional). *The function $p : C(\Omega) \rightarrow \mathbb{R}$ given by $p(f) = \log \lambda_f$ is called pressure functional.*

In Thermodynamic Formalism what is usually called pressure functional is a function $P : C(\Omega) \rightarrow \mathbb{R}$ given by

$$P(f) \equiv \sup_{\mu \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})} \{h(\mu) + \int_{\Omega} f d\mu\}.$$

After developing some perturbation theory we will show latter that both definitions of the Pressure functional are equivalent for any continuous potential, i.e., $P = p$.

Since Ω is compact and the space of all γ -Hölder continuous function $C^{\gamma}(\Omega)$ is an algebra of functions that separate points and contain the constant functions, we can apply the Stone-Weierstrass theorem to conclude that the closure of $C^{\gamma}(\Omega)$ in the uniform topology is $C(\Omega)$. Therefore for any arbitrary continuous potential f there is a sequence $(f_n)_{n \in \mathbb{N}}$ of Hölder continuous potentials such that $\|f_n - f\|_{\infty} \rightarrow 0$, when $n \rightarrow \infty$. For such uniform convergent sequences we will see that $p(f_n)$ converges to $p(f)$. In fact, a much stronger result can be stated. The pressure functional is Lipschitz continuous function on the space $C(\Omega)$.

Proposition 2. *If $f, g : \Omega \rightarrow \mathbb{R}$ are two arbitrary continuous potentials then $|p(f) - p(g)| \leq \|f - g\|_{\infty}$.*

Proof. The proof is an immediate consequence of the Proposition 1 and the inequality (6). \square

Corollary 2. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of continuous potentials such that $f_n \rightarrow f$ uniformly, then $p(f_n) \rightarrow p(f)$. In particular, $\lambda_{f_n} \rightarrow \lambda_f$.*

4 DLR-Gibbs Measures and Eigenmeasures

In this section we discuss the concept of specifications in the Thermodynamic Formalism setting. Some of its elementary properties for finite state space is discussed in details within this framework in the reference [8].

For each $n \in \mathbb{N}$, we define the projection on the n -th coordinate $\pi_n : \Omega \rightarrow M$ by $\pi_n(x) = x_n$. We use the notation \mathcal{F}_n to denote the sigma-algebra generated by the projections π_1, \dots, π_n . On the other hand, the notation $\sigma^n(\mathcal{F})$ stands for the sigma-algebra generated by the collection of projections $\{\pi_k : k \geq n+1\}$.

Let $f \in C(\Omega)$ a potential and for each $n \in \mathbb{N}$, $x \in \Omega$ and $E \in \mathcal{F}$ consider the mapping $K_n : \mathcal{F} \times \Omega \rightarrow [0, 1]$ given by

$$K_n(E, x) \equiv \frac{\mathcal{L}_f^n(1_E)(\sigma^n(x))}{\mathcal{L}_f^n(1)(\sigma^n(x))}. \quad (7)$$

For any fixed $x \in \Omega$ follows from the monotone convergence theorem that the map $\mathcal{F} \ni E \mapsto K_n(E, x)$ is a probability measure. For any fixed measurable set $E \in \mathcal{F}$ follows from the Fubini theorem that the map $x \mapsto K_n(E, x)$ is $\sigma^n(\mathcal{F})$ -measurable. So K_n is a **probability Kernel** from $\sigma^n(\mathcal{F})$ to \mathcal{F} .

Notice for any $\varphi \in C(\Omega)$ that $K_n(\varphi, x)$ is naturally defined because the rhs of (7). It is easy to see (using rhs of (7)) that they are proper kernels, meaning that for any bounded $\sigma^n(\mathcal{F})$ -measurable function φ we have $K_n(\varphi, x) = \varphi(\sigma^n(x))$. The above probability kernels have the following important property. For any fixed continuous function φ the map $x \mapsto K_n(\varphi, x)$ is continuous as consequence of Lebesgue dominated convergence theorem. We refer to this saying that $(K_n)_{n \in \mathbb{N}}$ has the **Feller Property**.

Definition 3. A Gibbsian specification with parameter set \mathbb{N} in the translation invariant setting is an abstract family of probability Kernels $K_n : (\mathcal{F}, \Omega) \rightarrow [0, 1]$, $n \in \mathbb{N}$ such that

- a) $\Omega \ni x \mapsto K_n(E, x)$ is $\sigma^n \mathcal{F}$ -measurable function for any $E \in \mathcal{F}$;
- b) $\mathcal{F} \ni E \mapsto K_n(E, x)$ is a probability measure for any $x \in \Omega$;
- c) for any $n, r \in \mathbb{N}$ and any bounded \mathcal{F} -measurable function $f : \Omega \rightarrow \mathbb{R}$ we have **the compatibility condition**, i.e.,

$$K_{n+r}(f, x) = \int_{\Omega} K_n(f, \cdot) dK_{n+r}(\cdot, x) \equiv K_{n+r}(K_n(f, \cdot), x).$$

Remark 1. The classical definition of a specification as given in [12] requires even in our setting a larger family of probability kernels. To be more precise we have to define a probability kernel for any finite subset $\Lambda \subset \mathbb{N}$ and the kernels K_Λ have to satisfy a), b) and a generalization of c). In translation invariant setting on the lattice \mathbb{N} the formalism can be simplified and one needs only to consider the family K_n , $n \in \mathbb{N}$, as defined above. Strictly speaking to be able to use the results in [12] one has first to

extend our specifications to any set $\Lambda = \{n_1, \dots, n_r\}$, but this can be consistently done by putting $K_\Lambda \equiv K_{n_r}$. This simplified definition adopted here is further justified by the fact that the DLR-Gibbs measures, compatible with a specification with parameter set \mathbb{N} , are completely determined by the kernels indexed in any cofinal collection of subsets of \mathbb{N} . So here we are taking advantage of this result to define our kernels only on the cofinal collection of subsets of \mathbb{N} of the form $\{1, \dots, n\}$ with $n \in \mathbb{N}$. Therefore when we write K_n we are really thinking, in terms of the general definition of specifications, about $K_{\{1, \dots, n\}}$.

The only specifications needed here are the ones described by (7), which is defined in terms of any continuous potential f . Notice that in the translation invariant setting the construction in (7), for the lattice \mathbb{N} , extends the usual construction made in terms of regular interactions. But in any case (7) give us particular constructions of quasilocal specifications which allow us to use some of the results from [12]. We refer the reader to [8] for results about specifications on the Ruelle operator setting.

Using the same ideas employed in the proof of Theorem 23 in [8] one can prove for any $r, n \in \mathbb{N}$, $x \in \Omega$ and $\varphi \in C(\Omega)$ the following identity

$$\mathcal{L}_f^{n+r}(\varphi)(\sigma^{n+r}(x)) = \mathcal{L}_f^{n+r} \left(\frac{\mathcal{L}_f^n(\varphi)(\sigma^n(\cdot))}{\mathcal{L}_f^n(1)(\sigma^n(\cdot))} \right)(\sigma^{n+r}(x)). \quad (8)$$

The above identity immediately implies for the Kernels defined by (7) that

$$K_{n+r}(f, x) = \int_{\Omega} K_n(f, \cdot) dK_{n+r}(\cdot, x) \equiv K_{n+r}(K_n(f, \cdot), x). \quad (9)$$

As we mentioned before, we refer to the above set of identities as compatibility conditions for the family of probability kernels $(K_n)_{n \in \mathbb{N}}$ or simply DLR equations.

Definition 4. We say that $\mu \in \mathcal{P}(\Omega, \mathcal{F})$ is a DLR-Gibbs measure (or, just DLR) for the continuous potential f if for any n and any continuous function $\varphi : \Omega \rightarrow \mathbb{R}$ we have for μ -almost all x that

$$\mathbb{E}_\mu[\varphi | \sigma^n(\mathcal{F})](x) = \int \varphi(y) dK_n(y, x).$$

The set of all DLR-Gibbs measures for f is denoted by $\mathcal{G}^{DLR}(f)$.

One very important and elementary result on DLR-Gibbs measure is the equivalence between the two conditions below:

- a) $\mu \in \mathcal{G}^{DLR}(f)$;
- b) for any $n \in \mathbb{N}$ and $E \in \mathcal{F}$ we have that $\mu(E) = \int_{\Omega} K_n(E, \cdot) d\mu$.

As usual to prove that $\mu \in \mathcal{G}^{DLR}(f)$ is not empty one uses the following result.

Lemma 1. *For any $f \in C(\Omega)$ the closure of the convex hull of the weak limits of the form $K_n(\cdot, x)$, where x run over all elements of Ω is equal to $\mathcal{G}(f)$.*

Proof. As it is stated the proof of this lemma can be found Theorem 29 in [8] with the suitable adaptations it works the same on general compact state space M . We should remark that this result is valid in much more generality for quasilocal specifications and its proof is presented in [12, p. 67]. \square

Lemma 2. *Let $f \in C(\Omega)$ be a potential and $(K_n)_{n \in \mathbb{N}}$ the specification defined by (7). Then we have $\mathcal{G}^*(f) \subset \mathcal{G}^{DLR}(f)$.*

Proof. Let ν be such that $\mathcal{L}_f^* \nu = \lambda_f \nu$ and φ a bounded \mathcal{F} -measurable function. Notice that the quotient appearing in the first integral below is $\sigma^n(\mathcal{F})$ -measurable. Therefore for any bounded \mathcal{F} -measurable ψ the following equality holds.

$$\begin{aligned} \int_{\Omega} (\varphi \circ \sigma^n)(x) \frac{\mathcal{L}_f^n(\psi)(\sigma^n(x))}{\mathcal{L}_f^n(1)(\sigma^n(x))} d\nu(x) &= \int_{\Omega} \frac{\mathcal{L}_f^n(\psi(\varphi \circ \sigma^n))(\sigma^n(x))}{\mathcal{L}_f^n(1)(\sigma^n(x))} d\nu(x) \\ &= \int_{\Omega} \frac{1}{\lambda^n} \mathcal{L}_f^n \left[\frac{\mathcal{L}_f^n(\psi(\varphi \circ \sigma^n))(\sigma^n(\cdot))}{\mathcal{L}_f^n(1)(\sigma^n(\cdot))} \right] (x) d\nu(x). \end{aligned}$$

By using the equation (8) we see that rhs above is equals to

$$\int_{\Omega} \frac{1}{\lambda^n} \mathcal{L}_f^n(\psi(\varphi \circ \sigma^n))(x) d\nu(x) = \int_{\Omega} \psi(x) (\varphi \circ \sigma^n)(x) d\nu(x).$$

Since φ is an arbitrary \mathcal{F} -measurable function we can conclude that

$$\nu[E|\sigma^n \mathcal{F}](y) = \frac{\mathcal{L}_f^n(I_E)(\sigma^n(y))}{\mathcal{L}_f^n(1)(\sigma^n(y))} \quad \nu - \text{a.s.}$$

so the equation (7) implies that $\nu \in \mathcal{G}(f)$. \square

The next lemma establishes the reverse inclusion between the set $\mathcal{G}^{DLR}(f)$ and the set $\mathcal{G}^*(f)$ of eigenprobabilities for the dual of the Ruelle operator. Its proof is much more involved than previous one and before proceed we recall some classical results about Martingales and Specification Theory which we will be used in the sequel.

Theorem A (Backward Martingale Convergence Theorem). *Consider the following sequence of σ -algebras $\mathcal{F} \supset \sigma \mathcal{F} \supset \dots \supset \cap_{n \in \mathbb{N}} \sigma^n \mathcal{F}$ and a bounded \mathcal{F} -measurable function $\varphi : \Omega \rightarrow \mathbb{R}$. Then, for any $\mu \in \mathcal{P}(\Omega, \mathcal{F})$ we have*

$$\mu[\varphi|\sigma^n \mathcal{F}] \rightarrow \mu[\varphi|\cap_{j=1}^{\infty} \sigma^j \mathcal{F}], \quad \text{a.s. and in } L^1(\Omega, \mathcal{F}, \mu).$$

Theorem B. *Let $(K_n)_{n \in \mathbb{N}}$ be the specification given in (7). Then the following conclusion holds. A probability measure $\mu \in \mathcal{G}^{DLR}(f)$ is extreme in $\mathcal{G}^{DLR}(f)$, if and only if, μ is trivial on $\cap_{n \in \mathbb{N}} \sigma^n \mathcal{F}$. As consequence if μ is extreme in $\mathcal{G}^{DLR}(f)$, then every $\cap_{n \in \mathbb{N}} \sigma^n \mathcal{F}$ -measurable function f is constant μ a.s..*

We give a proof of the above result in our setting in the appendix. From now on we eventually refer to $\cap_{n \in \mathbb{N}} \sigma^n \mathcal{F}$ as the tail sigma-algebra. Now we are ready to prove one of the main results of this section.

Lemma 3. *Let $f \in C(\Omega)$ be a potential and $(K_n)_{n \in \mathbb{N}}$ defined as in (7). Then we have $\mathcal{G}^{DLR}(f) \subset \mathcal{G}^*(f) = \{\nu \in \mathcal{P}(\Omega, \mathcal{F}) : \mathcal{L}_f^* \nu = \lambda_f \nu\}$.*

Proof. Since $\mathcal{P}(\Omega, \mathcal{F})$ is compact if we prove that any extreme element $\mu \in \mathcal{G}^{DLR}(f)$ is an eigenprobability of \mathcal{L}_f^* associated to λ_f then will follow from the Krein-Milman theorem that $\mathcal{G}^{DLR}(f) \subset \{\nu \in \mathcal{P}(\Omega, \mathcal{F}) : \mathcal{L}_f^* \nu = \lambda_f \nu\}$.

Let $\mu \in \mathcal{G}^{DLR}(f)$ be an extreme element, $\varphi \in C(\Omega)$ and λ_f the eigenvalue of \mathcal{L}_f^* . From the elementary properties of the conditional expectation and definition of $\mathcal{G}(f)$ we have

$$\lambda_f \int_{\Omega} \varphi d\mu = \lambda_f \int_{\Omega} \mathbb{E}_{\mu}[\varphi | \sigma^{n+1} \mathcal{F}] d\mu = \lambda_f \int_{\Omega} \frac{\mathcal{L}_f^{n+1}(\varphi)(\sigma^{n+1}(x))}{\mathcal{L}_f^{n+1}(1)(\sigma^{n+1}(x))} d\mu$$

Now we rewrite the above integrand as follows

$$\lambda_f \cdot \frac{\mathcal{L}_f^{n+1}(\varphi)(\sigma^{n+1}(x))}{\mathcal{L}_f^{n+1}(1)(\sigma^{n+1}(x))} = \frac{\mathcal{L}_f^n(\mathcal{L}_f \varphi)(\sigma^n(\sigma x))}{\mathcal{L}_f^n(1)(\sigma^n(\sigma x))} \cdot \lambda_f \frac{\mathcal{L}_f^n(1)(\sigma^{n+1}(x))}{\mathcal{L}_f^{n+1}(1)(\sigma^{n+1}(x))}.$$

The above equation because of the definition of $\mathcal{G}^{DLR}(f)$ can be rewritten μ a.s. as follows

$$\begin{aligned} \lambda_f \mathbb{E}_{\mu}[\varphi | \sigma^{n+1} \mathcal{F}](x) &= \mathbb{E}_{\mu}[\mathcal{L}_f \varphi | \sigma^n \mathcal{F}](\sigma x) \cdot \lambda_f \frac{\mathcal{L}_f^n(1)(\sigma^{n+1}(x))}{\mathcal{L}_f^{n+1}(1)(\sigma^{n+1}(x))} \\ &\equiv \mathbb{E}_{\mu}[\mathcal{L}_f \varphi | \sigma^n \mathcal{F}] \circ \sigma(x) \cdot \lambda_f \frac{\mathcal{L}_f^n(1)(\sigma^{n+1}(x))}{\mathcal{L}_f^{n+1}(1)(\sigma^{n+1}(x))}. \end{aligned}$$

By using the Backward Martingale Convergence Theorem and μ -triviality of $\cap_{n \in \mathbb{N}} \sigma^n \mathcal{F}$ -measurable functions (see Corollary 5 in the Appendix) we can ensure that there exist the limit below and it is μ a.s. constant

$$\lim_{n \rightarrow \infty} \lambda_f \frac{\mathcal{L}_f^n(1)(\sigma^{n+1}(x))}{\mathcal{L}_f^{n+1}(1)(\sigma^{n+1}(x))} \equiv \varrho_f.$$

From the two above identities and the Backward Martingale Convergence Theorem we have μ a.s. point x

$$\lambda_f \mathbb{E}_{\mu}[\varphi | \cap_{n \in \mathbb{N}} \sigma^n \mathcal{F}](x) = \mathbb{E}_{\mu}[\mathcal{L}_f \varphi | \cap_{n \in \mathbb{N}} \sigma^n \mathcal{F}] \circ \sigma(x) \cdot \varrho_f.$$

Since μ is extreme in $\mathcal{G}^{DLR}(f)$ it follows that the r.v. $\mathbb{E}_{\mu}[\mathcal{L}_f \varphi | \cap_{n \in \mathbb{N}} \sigma^n \mathcal{F}]$ is constant μ a.s. and one version of this conditional expectation is given by the constant function $\int_{\Omega} \mathcal{L}_f(\varphi) d\mu$. Using this information and taking expectations on both sides above we get

$$\lambda_f \int_{\Omega} \varphi d\mu = \varrho_f \int_{\Omega} \mathcal{L}_f(\varphi) d\mu.$$

Therefore $\mathcal{L}_f^*(\mu) = (\lambda_f/\varrho_f)\mu$. At this point we already proved that μ is an eigenprobability, but more stronger result can be shown, which is, $\varrho_f = 1$ and so μ is eigenprobability associated to λ_f the maximal eigenvalue. Indeed, by proceeding as in the proof of Proposition 1 we can see that

$$\mathcal{L}_f^n(1)(\sigma^n x) \leq \exp(n\varepsilon) \int_{\Omega} \mathcal{L}_f^n(1)(y) d\mu(y) = \exp(n\varepsilon) \frac{\lambda_f^n}{\varrho_f^n}.$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_f^n(1)(\sigma^n x) \leq \varepsilon + \log \lambda_f - \log \varrho_f$$

Similar lower bounds can be shown, so the Proposition 1 implies that $\log \varrho_f = 0$ which implies that $\varrho_f = 1$ and the lemma follows. \square

The two previous lemmas give us the following theorem.

Theorem 2. *Let $f \in C(\Omega)$ be a potential and $(K_n)_{n \in \mathbb{N}}$ defined as in (7). Then, we have $\mathcal{G}^{DLR}(f) = \mathcal{G}^*(f) = \{\nu \in \mathcal{P}(\Omega, \mathcal{F}) : \mathcal{L}_f^* \nu = \lambda_f \nu\}$.*

5 Quasilocality of K_n

Definition 5 (Local Function). *A function $f : \Omega \rightarrow \mathbb{R}$ is said to be local if f is \mathcal{F}_n -measurable for some $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we denote by L_n the space of all **bounded** \mathcal{F}_n -measurable local functions and $L = \cup_{n \in \mathbb{N}} L_n$ the set of all bounded local functions.*

Remark 2. *Note that if M is finite, then any local function is continuous. Indeed, any element $f \in L_n$ is a function depending only on the first n coordinates.*

Definition 6 (Quasilocal Function). *A function $f : \Omega \rightarrow \mathbb{R}$ is said to be a **quasilocal** function if there is a sequence $(f_n)_{n \in \mathbb{N}}$ in L such that $\|f - f_n\|_{\infty} \rightarrow 0$, when $n \rightarrow \infty$. Here $\|\cdot\|_{\infty}$ is the sup-norm. We write \bar{L} to denote the space of all **bounded** quasilocal functions.*

Definition 7. *We say that the specification $(K_n)_{n \in \mathbb{N}}$ is quasilocal, if for each $n \in \mathbb{N}$ and $\varphi \in \bar{L}$ the mapping $x \mapsto K_n(\varphi, x)$ is quasilocal.*

Proposition 3. *For each $f \in C(\Omega)$ there is a sequence $(f_n)_{n \in \mathbb{N}}$ such that f_n is continuous, local (more precisely \mathcal{F}_n -measurable) and $\|f_n - f\|_{\infty} \rightarrow 0$, when $n \rightarrow \infty$.*

Proof. We fix a point in M , for sake of simplicity, this point will be denoted by 0. For each $n \in \mathbb{N}$ we define $f_n : \Omega \rightarrow \mathbb{R}$ by $f_n(x) = f(x_1, \dots, x_n, 0, 0, \dots)$. Since Ω is compact the function f is uniformly continuous, so for any given $\varepsilon > 0$ there is $\delta > 0$ such that for every pair $x, y \in \Omega$ satisfying $d_{\Omega}(x, y) < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Notice that for any $x \in \Omega$ we have

$$d_{\Omega}(x, (x_1, \dots, x_n, 0, 0, \dots)) = \sum_{j=n+1}^{\infty} \frac{1}{2^j} d(x_j, 0) \leq \text{diam}(\Omega) \sum_{j=n+1}^{\infty} \frac{1}{2^j} = \frac{\text{diam}(\Omega)}{2^n}.$$

If n is such that $\text{diam}(\Omega)2^{-n} < \delta$, then for all $x \in \Omega$ we have

$$|f(x) - f_n(x)| = |f(x) - f(x_1, \dots, x_n, 0, 0, \dots)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and the last inequality is uniform in x we have $\|f_n - f\|_\infty \rightarrow 0$, when $n \rightarrow \infty$. For each $n \in \mathbb{N}$ we clearly have that f_n is continuous, bounded and depends only on the first n coordinates and thus an element of L_n . \square

Theorem 3. *For any fixed potential $f \in C(\Omega)$ and $n \in \mathbb{N}$, we have that the Kernel K_n as defined in (7) is quasilocal.*

Proof. Let φ a quasilocal function and $(\varphi_n)_{n \in \mathbb{N}}$ a sequence of bounded local functions converging uniformly to φ . From the definition of K_n and the Ruelle operator we have $K_n(\varphi, x)$ is equal to

$$\frac{\int_{M^n} \exp(S_n(f)(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots)) \varphi(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots) \prod_{j=1}^n dp(a_j)}{\int_{M^n} \exp(f(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots)) \prod_{j=1}^n dp(a_j)},$$

where $S_n(f)(x) \equiv f(x) + f(\sigma x) + \dots + f(\sigma^{n-1}x)$.

For each $m \in \mathbb{N}$ define $\psi_m : \Omega \rightarrow \mathbb{R}$, where $\psi_m(x)$, for any $x \in \Omega$, is given the following expression

$$\frac{\int_{M^n} \exp(S_n(f_m)(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots)) \varphi_m(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots) \prod_{j=1}^n dp(a_j)}{\int_{M^n} \exp(S_n(f_m)(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots)) \prod_{j=1}^n dp(a_j)},$$

Since $S_n(f_m)$ and φ_m are local, it follows that ψ_m is local. Clearly ψ_m is bounded.

We claim that for each fixed $n \in \mathbb{N}$ we have $\|\psi_m - K_n(\varphi, \cdot)\|_\infty \rightarrow 0$, when $m \rightarrow \infty$. Since n is fixed, we can introduce a more convenient notation

$$W(g)(a, x) = \exp(S_n(g)(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots)).$$

By using that $\|f_m - f\|_\infty \rightarrow 0$, when $m \rightarrow \infty$, we get for each fixed $(a_1, \dots, a_n) \in E^n$ that

$$\lim_{m \rightarrow \infty} \sup_{x \in \Omega} |W(f_m)(a, x) - W(f)(a, x)| = 0. \quad (10)$$

Using the W notation ψ_m has simpler expression

$$\psi_m(x) = \frac{\int_{M^n} W(f_m)(a, x) \varphi_m(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots) \prod_{j=1}^n dp(a_j)}{\int_{M^n} W(f_m)(a, x) \prod_{j=1}^n dp(a_j)}$$

It is easy to see that $\exp(-n\|f\|_\infty) \leq |W(f_m)(a, x)| \leq \exp(n\|f\|_\infty)$ and $\|\varphi_m\|_\infty \leq 1 + \|\varphi\|_\infty$ for m large enough. From these estimates and the Dominated Convergence Theorem we have have

$$\lim_{m \rightarrow \infty} \psi_m(x) = \lim_{m \rightarrow \infty} \frac{\int_{M^n} W(f_m)(a, x) \varphi_m(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots) \prod_{j=1}^n dp(a_j)}{\int_{M^n} W(f_m)(a, x) \prod_{j=1}^n dp(a_j)}$$

$$\begin{aligned}
&= \frac{\int_{M^n} W(f)(a, x) \varphi(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots) \prod_{j=1}^n dp(a_j)}{\int_{M^n} W(f)(a, x) \prod_{j=1}^n dp(a_j)} \\
&= K_n(\varphi, x).
\end{aligned}$$

Now we prove that the convergence $|\psi_m(x) - K_n(\varphi, x)| \rightarrow 0$, when $m \rightarrow \infty$ is uniform in x . Indeed, by using that $\|\varphi_m - \varphi\|_\infty \rightarrow 0$, when $m \rightarrow \infty$ and (10) we have for any fixed a_1, \dots, a_n that

$$\lim_{m \rightarrow \infty} \sup_{x \in \Omega} |W(f_m)(a, x) \varphi_m(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots) - W(f)(a, x) \varphi(a_1, \dots, a_n, x_{n+1}, x_{n+2}, \dots)| = 0. \quad (11)$$

The above limit implies that the numerator of ψ_m converges uniformly to the numerator of $K_n(\varphi, \cdot)$. The expression 10 guarantees that the denominator of ψ_m converges uniformly to the denominator of $K_n(\varphi, \cdot)$, thus proving the theorem. \square

6 Strongly Non-null Specifications

Let us assume that the state space M is a finite set of the form $M = \{1, 2, \dots, d\}$. We recall that in the general theory of specifications, we say that a specification $\gamma = (\gamma_\Lambda)_{\Lambda \subset S}$ with parameter set S on (M^S, \mathcal{F}) is **strongly non-null** if there exist a constant c such that for all $i \in S$ and $\omega \in \Omega$ we have $\gamma_{\{i\}}(\omega_i | \cdot) \geq c > 0$, see [12, 11] for more details on specifications.

For our particular specification $(K_n)_{n \in \mathbb{N}}$ the above condition is simply

$$K_i(\mathcal{C}_i(a), \cdot) \equiv \frac{\mathcal{L}_f^i(1_{\mathcal{C}_i(a)})(\sigma^i(\cdot))}{\mathcal{L}_f^i(1)(\sigma^i(\cdot))} > c > 0,$$

where $a \in \{1, 2, \dots, d\}$ and $\mathcal{C}_i(a) = \{\omega \in \Omega : \omega_i = a\}$ is cylinder set.

Specifications Associated to Hölder Potentials are Strongly Non-null

Let $\Omega \equiv \{1, 2, \dots, d\}^{\mathbb{N}}$, $f : \Omega \rightarrow \mathbb{R}$ be a normalized Hölder potential, i.e.,

$$\mathcal{L}_f(1)(x) \equiv 1, \quad \forall x \in \Omega,$$

and $(K_n)_{n \in \mathbb{N}}$ the specification (7) defined by f . We fix $a \in \{1, 2, \dots, d\}$ and $x = (x_1, x_2, \dots) \in \Omega$. From the definition of the specification $(K_n)_{n \in \mathbb{N}}$ it follows that

$$\begin{aligned}
K_i(\mathcal{C}_i(a), (x_1, x_2, x_3, \dots)) &= \frac{\mathcal{L}_f^i(1_{\mathcal{C}_i(a)})(\sigma^i(x_1, x_2, \dots))}{\mathcal{L}_f^i(1)(\sigma^i(x_1, x_2, \dots))} = \mathcal{L}_f^i(1_{\mathcal{C}_i(a)})(\sigma^i(x_1, x_2, \dots)) \\
&= \mathcal{L}_f^{i-1}(1)(a, x_{i+1}, x_{i+2}, \dots) e^{f(a, x_{i+1}, x_{i+2}, \dots)}.
\end{aligned}$$

Since $e^{f(a, x_{i+1}, x_{i+2}, \dots)}$ is uniformly bounded away from zero and

$$\mathcal{L}_f^{i-1}(1)(a, x_{i+1}, x_{i+2}, \dots) = 1,$$

then the specification $(K_n)_{n \in \mathbb{N}}$ associated to f satisfies the strongly non-null condition.

Now we consider the case of a general Hölder potential f . Let $h_f > 0$ the main eigenfunction of the Ruelle operator \mathcal{L}_f , associated to the main eigenvalue λ_f . We assume that the equilibrium state μ for f satisfies $\int_{\Omega} h_f d\mu = 1$. We denote by \bar{f} the associated normalized potential, i.e., $\bar{f} = f + \log h_f - \log \circ h_f \circ \sigma - \log \lambda_f$.

Now, for any fixed $a \in \{1, 2, \dots, d\}$ and $x = (x_1, x_2, \dots) \in \Omega$ we get

$$\begin{aligned} K_i(\mathcal{C}_i(a), (x_1, x_2, x_3, \dots)) &= \frac{\mathcal{L}_f^i(1_{\mathcal{C}_i(a)})(\sigma^i(x_1, x_2, \dots))}{\mathcal{L}_f^i(1)(\sigma^i(x_1, x_2, \dots))} \\ &= \frac{\mathcal{L}_{\bar{f}}^{i-1}(h_f^{-1})(a, x_{i+1}, x_{i+2}, \dots) h_f(a, x_{i+1}, x_{i+2}, \dots) \lambda_f^i}{\mathcal{L}_f^i(1)(x_{i+1}, x_{i+2}, \dots)}. \end{aligned}$$

It is known that h_f is uniformly bounded away from zero and

$$\lim_{i \rightarrow \infty} \mathcal{L}_{\bar{f}}^{i-1}(h_f^{-1})(a, x_{i+1}, x_{i+2}, \dots) = \int_{\Omega} h_f^{-1} d\mu > 0.$$

Moreover, we have that

$$\|\lambda_f^{-i} \mathcal{L}_f^i(1) - h_f\|_{\infty} \xrightarrow{n \rightarrow \infty} 0,$$

thus proving that for any choice of $a \in \{1, 2, \dots, d\}$ that $K_i(\mathcal{C}_i(a), x)$ is uniformly bounded away from zero, in both $x \in \Omega$ and $i \in \mathbb{N}$.

Breaking the Strongly Non-null Condition - Double Hofbauer

We now present an example of a continuous potential, called Double Hofbauer, for which the associated specification does not satisfy the strongly non-null condition. The Double Hofbauer potential is defined on the symbolic space $\Omega = \{0, 1\}^{\mathbb{N}}$ as follows: for each $n \geq 1$ we consider the following cylinder subsets of Ω

$$L_n = \underbrace{000 \dots 0}_n 1 \quad \text{and} \quad R_n = \underbrace{111 \dots 1}_n 0, \quad \text{for all } n \geq 1. \quad (12)$$

Note that these cylinders are disjoint and $(\cup_{n \geq 1} L_n) \cup (\cup_{n \geq 1} R_n) = \Omega \setminus \{0^{\infty}, 1^{\infty}\}$. We fix two real numbers $\gamma > 1$ and $\delta > 1$, satisfying $\delta < \gamma$ and the Double Hofbauer potential f is given by the following expression:

$$f(x) = \begin{cases} -\gamma \log \frac{n}{n-1}, & \text{if } x \in L_n, \text{ for some } n \geq 2; \\ -\delta \log \frac{n}{n-1}, & \text{if } x \in R_n, \text{ for some } n \geq 2; \\ -\log \zeta(\gamma), & \text{if } x \in L_1; \\ -\log \zeta(\delta), & \text{if } x \in R_1; \\ 0, & \text{if } x \in \{1^{\infty}, 0^{\infty}\}, \end{cases}$$

where $\zeta(s) = \sum_{n \geq 1} 1/n^s$. Note that this is a continuous potential and below we have a sketch of its graph

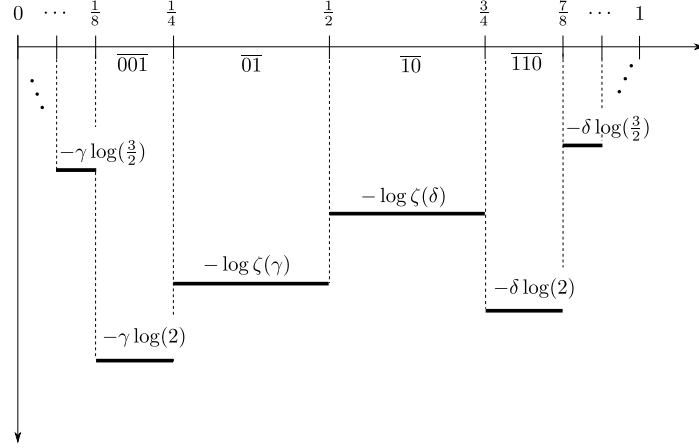


Figure 1: The Double Hofbauer Potential represented in the closed interval $[0, 1]$.

We refer the reader to [9] for the analysis of phase transition in this model and some facts about this model that will be used in the sequel.

The first important fact we use next is that the Ruelle operator associated to this potential has a non-negative eigenfunction h_f which is continuous in almost all points. An explicit description of this eigenfunction is given in section 4 in [9].

Nevertheless we can define a normalized potential

$$\bar{f} = \log J = f + \log h_f - \log(h_f \circ \sigma) - \log \lambda_f,$$

which is only discontinuous at a finite number of points.

We observe that $\exp(\bar{f}) = J$ is continuous, although $\bar{f} = \log J$ can attain the value $-\infty$. The continuous function J , can take the values 0 and 1 on a finite number of points, see item f) on the beginning of the proof of Proposition 29 in [9]. Therefore $\mathcal{L}_{\bar{f}} = \mathcal{L}_{\log J}$ defines a positive bounded linear operator acting on $C(\Omega)$.

Let us proceed to analyze, first, the strongly non-null condition for \bar{f} (which is not continuous). In order to prove that the strongly non-null condition is broken we take $a = 0$ and $x = 1^\infty$. In the same way as before we get

$$K_i(\mathcal{C}_i(0), (1, 1, 1, \dots)) = \mathcal{L}_{\bar{f}}^{i-1}(1)(0, 1, 1, \dots) \exp(\bar{f}(0, 1, 1, \dots)).$$

From items e) and f) on the explicit computation of $\exp(\bar{f})$ appearing in Proposition 29 of [9], we get that $\exp(\bar{f}(0, 1, 1, \dots)) = 0$.

From the computation on page 27 in [9] follows that

$$\lim_{i \rightarrow \infty} \frac{\mathcal{L}_{\bar{f}}^i(1_{\mathcal{C}_i(0)})(\sigma^i(1^\infty))}{\mathcal{L}_{\bar{f}}^i(1)(\sigma^i(1^\infty))} = 0.$$

Thus showing that the specification associated to the potential $\bar{f} = \log J$ does **not** satisfy the strongly non-null condition.

Now we turn attention to the Double Hofbauer potential f . Before to present computations for f we will need to some expressions evolving its normalization \bar{f} . As we observed above $\exp(\bar{f}) = J$ is continuous mapping so the duality relation is well defined and we have

$$\int_{\Omega} \varphi d[\mathcal{L}_{\bar{f}}^* \nu] = \int_{\Omega} \mathcal{L}_{\bar{f}} \varphi d\nu, \quad \forall \varphi \in C(\Omega).$$

This allow us to define for each $n \geq 1$ and $y \in \Omega$ the following probability measure on Ω

$$\mu_n^y = [(\mathcal{L}_{\bar{f}})^*]^n (\delta_{\sigma^n(y)}).$$

For a fixed $y \in \Omega$ any cluster point, with respect to the weak topology, of the sequence $(\mu_n^y)_{n \in \mathbb{N}}$, is called a thermodynamic limit with boundary condition y (see section 8 in [9]).

Let $\tilde{\mu}$ be any thermodynamic limit of the sequence $(\mu_n^{01^\infty})_{n \in \mathbb{N}}$, where the boundary condition $01^\infty \equiv (0, 1, 1, \dots)$. It was shown in [9] that (a true limit on i not a subsequence)

$$\lim_{i \rightarrow \infty} \mathcal{L}_{\log J}^i (1_{\mathcal{C}_1(0)})(0, 1, 1, \dots) = \int_{\Omega} 1_{\mathcal{C}_1(0)} d\tilde{\mu}.$$

We now back to the original potential (non-normalized Hofbauer), but considering further restrictions $\gamma, \delta > 2$. In this case it is known that the eigenvalue $\lambda_f = 1$. By using this fact, fixing $a = 0$ and proceeding in the same way as before, for any $x \in \Omega$, we get that

$$\begin{aligned} K_i(\mathcal{C}_i(0), (x_1, x_2, x_3, \dots)) &= \frac{\mathcal{L}_f^i(1_{\mathcal{C}_i(0)})(\sigma^i(x_1, x_2, \dots))}{\mathcal{L}_f^i(1)(\sigma^i(x_1, x_2, \dots))} \\ &= \frac{\mathcal{L}_{\log J}^{i-1}(h_f^{-1})(0, x_{i+1}, x_{i+2}, \dots) h_f(0, x_{i+1}, x_{i+2}, \dots)}{\mathcal{L}_f^i(1)(x_{i+1}, x_{i+2}, \dots)}. \end{aligned}$$

From now on $x = (1, 1, 1, \dots)$. It was shown in [9] that $h_f(0, 1, 1, \dots)$ is finite and non-zero. In Section 4 of [9] it is shown that h_f does not vanish. Notice that the cylinder $\mathcal{C}_1(0)$ is a countable union of cylinders sets of the form L_k (following the notation (12)) where $k \geq 1$. Since $\tilde{\mu}(\mathcal{C}_1(0)) > 0$, there exist $n \in \mathbb{N}$ such that $\tilde{\mu}(L_n) > 0$. This means that for some sequence $(i_k)_{k \in \mathbb{N}}$ we have

$$\lim_{k \rightarrow \infty} \mathcal{L}_{\log J}^{i_k} (1_{L_n})(0, 1, 1, \dots) > 0.$$

From the properties of h_f deduced in [9] follows that $0 < c \equiv \inf\{h_f(x) : x \in L_n\} < +\infty$. For any pair of functions $\varphi, \psi \in C(\Omega)$ satisfying $\varphi \geq \psi$, we have from the positivity of the Ruelle operator, that $\mathcal{L}_{\log J}^{i-1}(\varphi)(0, 1, 1, \dots) \geq \mathcal{L}_{\log J}^{i-1}(\psi)(0, 1, 1, \dots)$. By taking $\varphi \equiv h_f^{-1}$ and $\psi \equiv c^{-1} \cdot 1_{L_n}$ we immediately have

$$\lim_{i \rightarrow \infty} \mathcal{L}_{\log J}^{i-1}(h_f^{-1})(0, 1, 1, \dots) \geq \lim_{i \rightarrow \infty} \mathcal{L}_{\log J}^{i-1}(1_{L_n} c^{-1})(0, 1, 1, \dots) > 0.$$

On the other hand, $\lim_{i \rightarrow \infty} \mathcal{L}_f^i(1)(1, 1, 1, \dots) = \infty$. By using the estimates obtained above we can finally conclude that

$$\lim_{i \rightarrow \infty} \frac{\mathcal{L}_f^i(1_{\mathcal{C}_i(0)})(\sigma^i(1^\infty))}{\mathcal{L}_f^i(1)(\sigma^i(1^\infty))} = 0,$$

showing that the specification $(K_n)_{n \in \mathbb{N}}$ associated to the Double Hofbauer potential does **not** satisfy the strongly non-null condition.

7 Uniqueness Theorem for Eigenprobabilities

Theorem 4. *Let f be continuous potential and $(K_n)_{n \in \mathbb{N}}$ be the specification defined as in (7). Suppose that there is constant $c > 0$ such that for every cylinder set $F \in \mathcal{F}$ there is $n \in \mathbb{N}$ such that*

$$K_n(F, x) \geq cK_n(F, y)$$

for all $x, y \in \Omega$. Then, the set $\mathcal{G}^(f) = \{\nu \in \mathcal{P}(\Omega, \mathcal{F}) : \mathcal{L}_f^* \nu = \lambda_f \nu\}$ has only one element.*

Proof. Suppose that $\mathcal{G}^*(f) (= \mathcal{G}^{DLR}(f))$ contains two distinct elements μ and ν . Then the convex combination $(1/2)(\mu + \nu) \in \mathcal{G}^*(f) \setminus \text{ex}(\mathcal{G}^*(f))$, where $\text{ex}(\mathcal{G}^*(f))$ denotes the set of extremes measures of $\mathcal{G}^*(f)$. Therefore it is sufficient to show that $\mathcal{G}^*(f) \subset \text{ex}(\mathcal{G}^*(f))$.

Let $\mu \in \mathcal{G}^*(f)$, $E_0 \in \cap_{j \in \mathbb{N}} \sigma^j(\mathcal{F})$ and suppose that $\mu(E_0) > 0$. The existence of such set is ensured by the Theorem 7.7 item (c) in [12], which says that any element $\mu \in \mathcal{G}^{DLR}(f)$ is uniquely determined by its restriction to the tail σ -algebra $\cap_{j \in \mathbb{N}} \sigma^j(\mathcal{F})$ (see Corollary 5 in the Appendix). Since $\mu(E_0) > 0$ the probability measure $\nu \equiv \mu(\cdot | E_0) \in \mathcal{G}^{DLR}(f)$, see Theorem 7.7 (b) in [12] (or, see Corollaries 9 and 4 in the Appendix).

Let us prove that for all $E \in \mathcal{F}$ we have $\nu(E) \geq c\mu(E)$. Fix a cylinder set $F \in \mathcal{F}$ then for n big enough follows from the characterization of the DLR-Gibbs measures and from the hypothesis that

$$\begin{aligned} \nu(F) &= \int_{\Omega} K_n(F, x) d\nu(x) = \int_{\Omega} \left[\int_{\Omega} K_n(F, x) d\nu(x) \right] d\mu(y) \\ &\geq c \int_{\Omega} \left[\int_{\Omega} K_n(F, y) d\nu(x) \right] d\mu(y) \\ &= c \int_{\Omega} \left[\int_{\Omega} K_n(F, y) d\mu(y) \right] d\nu(x) \\ &= c\mu(F). \end{aligned}$$

Using the monotone class theorem we may conclude that for all $E \in \mathcal{F}$ we have $\nu(E) \geq c\mu(E)$. In particular, $0 = \nu(\Omega \setminus E_0) \geq c\mu(\Omega \setminus E_0)$ therefore $\mu(E_0) = 1$. Consequently μ is trivial on $\cap_{j \in \mathbb{N}} \sigma^j(\mathcal{F})$. Hence another application of Theorem 7.7 (a) of [12] (or, see Corollary 4) ensures that μ is extreme. \square

A consequence of this theorem is the following generalization to uncountable alphabets of the famous Bowen's condition, see [26].

Theorem 5. *Let f be a continuous potential and $S_n(f, x) \equiv f(x) + f(\sigma x) + \dots + f(\sigma^{n-1}x)$. If*

$$D \equiv \sup_{n \in \mathbb{N}} \sup_{\substack{x, y \in \Omega; \\ d(x, y) \leq 2^{-n}}} |S_n(f, x) - S_n(f, y)| < \infty$$

then the set $\mathcal{G}(f) = \{\nu \in \mathcal{P}(\Omega, \mathcal{F}) : \mathcal{L}_f^ \nu = \lambda_f \nu\}$ is a singleton.*

The proof as an easy consequence of the previous theorem and it was presented in details in [8] for finite state space M . The proof for general compact space M is identical.

This result generalize two conditions for uniqueness presented in two recent works by the authors when general compact state space M is considered, see [10] and [17]. In fact, the above theorem generalizes the Hölder, Walters (weak and stronger as introduced in [10]) and Bowen conditions because it can be applied for potentials defined on $\Omega = M^{\mathbb{N}}$, where the state space M is any general compact metric space.

8 The Extension of the Ruelle Operator to the Lebesgue Space $L^1(\Omega, \mathcal{F}, \nu_f)$

Let ν_f be the Borel probability measure obtained in previous sections and f any fixed continuous potential. In this section we show how to construct a bounded linear extension of the operator $\mathcal{L}_f : C(\Omega) \rightarrow C(\Omega)$ acting on $L^1(\Omega, \mathcal{F}, \nu_f)$, by abusing notation also called \mathcal{L}_f , and prove the existence of an almost surely non-negative eigenfunction $\varphi_f \in L^1(\Omega, \mathcal{F}, \nu_f)$ associated to the eigenvalue λ_f constructed in the previous section.

Proposition 4. *Fix a continuous potential f and let λ_f and ν_f be the eigenvalue and eigenmeasure of $\mathcal{L}_f^*/\mathcal{L}_f^*(1)$, respectively. Then the Ruelle operator $\mathcal{L}_f : C(\Omega) \rightarrow C(\Omega)$ can be uniquely extended to a bounded linear operator $\mathcal{L}_f : L^1(\Omega, \mathcal{F}, \nu_f) \rightarrow L^1(\Omega, \mathcal{F}, \nu_f)$ having its operator norm given by $\|\mathcal{L}_f\|_{L^1(\Omega, \mathcal{F}, \nu_f)} = \lambda_f$.*

Proof. If $\varphi \in C(\Omega)$ then $\varphi^\pm \equiv \max\{0, \pm\varphi\} \in C(\Omega)$. Therefore follows from the positivity of the Ruelle operator and (3) that

$$\begin{aligned} \|\mathcal{L}_f(\varphi)\|_{L^1} &= \int_{\Omega} |\mathcal{L}_f(\varphi^+ - \varphi^-)| d\nu_f \leq \int_{\Omega} |\mathcal{L}_f(\varphi^+)| + |\mathcal{L}_f(\varphi^-)| d\nu_f \\ &= \int_{\Omega} \mathcal{L}_f(\varphi^+) + \mathcal{L}_f(\varphi^-) d\nu_f = \int_{\Omega} (\varphi^+ + \varphi^-) d(\mathcal{L}_f^* \nu_f) \\ &= \lambda_f \int_{\Omega} (\varphi^+ + \varphi^-) d\nu_f = \lambda_f \int_{\Omega} |\varphi| d\nu_f \\ &= \lambda_f \|\varphi\|_{L^1}. \end{aligned}$$

Since Ω is a compact Hausdorff space we have

$$\overline{C(\Omega, \mathbb{R})}^{L^1(\Omega, \mathcal{F}, \nu_f)} = L^1(\Omega, \mathcal{F}, \nu_f),$$

therefore \mathcal{L}_f admits a unique continuous extension to $L^1(\Omega, \mathcal{F}, \nu_f)$. By taking $\varphi \equiv 1$ it is easy to see that $\|\mathcal{L}_f\|_{L^1(\Omega, \mathcal{F}, \nu_f)} = \lambda_f$. \square

Proposition 5. *For any fixed potential $f \in C(\Omega)$ we have that*

$$L^1(\Omega, \mathcal{F}, \nu_f) = \Xi(f) \equiv \left\{ \varphi \in L^1(\Omega, \mathcal{F}, \nu_f) : \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_f = \lambda_f \int_{\Omega} \varphi d\nu_f \right\}.$$

Proof. From (3) it follows that $C(\Omega) \subset \Xi(f)$. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence in $\Xi(f)$ such that $\varphi_n \rightarrow \varphi$ in $L^1(\Omega, \mathcal{F}, \nu_f)$. Then

$$\left| \int_{\Omega} \varphi_n d\nu_f - \int_{\Omega} \varphi d\nu_f \right| \leq \int_{\Omega} |\varphi_n - \varphi| d\nu_f \xrightarrow{n \rightarrow \infty} 0$$

and using the boundedness of \mathcal{L}_f , we can also conclude that

$$\left| \int_{\Omega} \mathcal{L}_f(\varphi_n) d\nu_f - \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_f \right| \leq \int_{\Omega} |\mathcal{L}_f(\varphi_n - \varphi)| d\nu_f \leq \lambda_f \|\varphi_n - \varphi\|_{L^1} \xrightarrow{n \rightarrow \infty} 0.$$

By using the above convergences and the triangular inequality we can see that $\Xi(f)$ is closed subset of $L^1(\Omega, \mathcal{F}, \nu_f)$. Indeed,

$$\begin{aligned} \left| \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_f - \lambda_f \int_{\Omega} \varphi d\nu_f \right| &\leq \\ \left| \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_f - \int_{\Omega} \mathcal{L}_f(\varphi_n) d\nu_f + \int_{\Omega} \mathcal{L}_f(\varphi_n) d\nu_f - \lambda_f \int_{\Omega} \varphi d\nu_f \right| \end{aligned}$$

and the rhs goes to zero when $n \rightarrow \infty$ therefore $\varphi \in \Xi(f)$. Since $C(\Omega, \mathbb{R}) \subset \Xi(f)$ and $\Xi(f)$ is closed in $L^1(\Omega, \mathcal{F}, \nu_f)$ we have that

$$L^1(\Omega, \mathcal{F}, \nu_f) = \overline{C(\Omega, \mathbb{R})}^{L^1(\Omega, \mathcal{F}, \nu_f)} \subset \overline{\Xi(f)}^{L^1(\Omega, \mathcal{F}, \nu_f)} = \Xi(f) \subset L^1(\Omega, \mathcal{F}, \nu_f).$$

\square

9 Strong Convergence of Ruelle Operators

Proposition 6. *For any fixed potential $f \in C(\Omega)$ there is a sequence $(f_n)_{n \in \mathbb{N}}$ contained in $C^\gamma(\Omega)$ such that $\|f_n - f\|_\infty \rightarrow 0$. Moreover, for any eigenmeasure ν_f associated to the eigenvalue λ_f we have that \mathcal{L}_{f_n} has a unique continuous extension to an operator defined on $L^1(\Omega, \mathcal{F}, \nu_f)$ and moreover in the uniform operator norm $\|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{L^1(\Omega, \mathcal{F}, \nu_f)} \rightarrow 0$, when $n \rightarrow \infty$.*

Proof. The first statement is a direct consequence of the Stone-Weierstrass Theorem.

For any $\varphi \in L^1(\Omega, \mathcal{F}, \nu_f)$ the extension of \mathcal{L}_{f_n} is given by $\mathcal{L}_{f_n}(\varphi) \equiv \mathcal{L}_f(\exp(f_n - f)\varphi)$ which is well-defined due to Proposition 4. From this proposition we can also get the following inequality

$$\begin{aligned} \int_{\Omega} |\mathcal{L}_{f_n}(\varphi)| d\nu_f &= \int_{\Omega} |\mathcal{L}_f(\exp(f_n - f)\varphi)| d\nu_f \\ &\leq \lambda_f \|\exp(f_n - f)\|_{\infty} \|\varphi\|_{L^1(\Omega, \mathcal{F}, \nu_f)} < \infty. \end{aligned}$$

Since the distance in the uniform operator norm between \mathcal{L}_{f_n} and \mathcal{L}_f can be upper bounded by

$$\begin{aligned} \|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{L^1(\Omega, \mathcal{F}, \nu_f)} &= \sup_{0 < \|\varphi\|_{L^1} \leq 1} \int_{\Omega} |\mathcal{L}_{f_n}(\varphi) - \mathcal{L}_f(\varphi)| d\nu_f \\ &\leq \sup_{0 < \|\varphi\|_{L^1} \leq 1} \int_{\Omega} |\mathcal{L}_f(\exp(f_n - f)\varphi) - \mathcal{L}_f(\varphi)| d\nu_f \\ &\leq \lambda_f \sup_{0 < \|\varphi\|_{L^1} \leq 1} \int_{\Omega} |\varphi| |(\exp(f_n - f) - 1)| d\nu_f \\ &\leq \lambda_f |\exp(\|f_n - f\|_{\infty}) - 1| \sup_{0 < \|\varphi\|_{L^1} \leq 1} \int_{\Omega} |\varphi| d\nu_f, \end{aligned}$$

we can conclude that $\|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{L^1(\Omega, \mathcal{F}, \nu_f)} \rightarrow 0$, when $n \rightarrow \infty$. □

10 Existence of the Eigenfunctions

We point out that given a continuous potential f there exist always eigenprobabilities for \mathcal{L}_f . However, does not always exist a positive continuous eigenfunction h for \mathcal{L}_f (see examples for instance in [9]).

In this section we consider sequences of Borel probability measures $(\mu_{f_n})_{n \in \mathbb{N}}$ defined by

$$\mathcal{F} \ni E \mapsto \mu_{f_n}(E) \equiv \int_E h_{f_n} d\nu_f, \quad (13)$$

where $f_n \in C^{\gamma}(\Omega)$ satisfies $\|f_n - f\|_{\infty} \rightarrow 0$, and h_{f_n} is the unique eigenfunction of \mathcal{L}_{f_n} , which is assumed to have $L^1(\Omega, \mathcal{F}, \nu_f)$ norm one. Since Ω is compact we can assume up to subsequence that $\mu_{f_n} \rightharpoonup \mu \in \mathcal{P}(\Omega, \mathcal{F})$.

We point out that for some parameters of the Hofbauer model $f : \{0, 1\} \rightarrow \mathbb{R}$, there exists a positive measurable eigenfunction h , but $\int h d\nu_f$ is not finite (see [9]).

From the definition of μ_{f_n} we immediately have that $\mu_{f_n} \ll \nu_f$. Notice that, in such generality, it is **not** possible to guarantee that $\mu \ll \nu_f$. When this is true the

Radon-Nikodym theorem ensures the existence of a non-negative function $d\mu/d\nu_f \in L^1(\Omega, \mathcal{F}, \nu_f)$ such that for all $E \in \mathcal{F}$ we have

$$\mu(E) = \int_E \frac{d\mu}{d\nu_f} d\nu_f. \quad (14)$$

In what follows we give sufficient conditions for this Radon-Nikodym derivative to be an eigenfunction of \mathcal{L}_f .

Theorem 6. *Let μ_{f_n} as in (13), f_n Holder approximating f . If $(h_{f_n})_{n \in \mathbb{N}}$ is a relatively compact subset of $L^1(\Omega, \mathcal{F}, \nu_f)$ then up to subsequence $\mu_{f_n} \rightharpoonup \mu$, $\mu \ll \nu_f$ and $\mathcal{L}_f(d\mu/d\nu_f) = \lambda_f d\mu/d\nu_f$.*

Proof. Without loss of generality we can assume that h_{f_n} converges to some non-negative function $h_f \in L^1(\Omega, \mathcal{F}, \nu_f)$. This convergence implies

$$\left| \int_{\Omega} \varphi h_{f_n} d\nu_f - \int_{\Omega} \varphi h_f d\nu_f \right| \rightarrow 0, \quad \forall \varphi \in C(\Omega).$$

Therefore $\mu_{f_n} \rightharpoonup \mu$ with $\mu \ll \nu_f$ and $d\mu/d\nu_f = h_f$ almost surely.

Let us show that this Radon-Nikodym derivative is a non-negative eigenfunction for the Ruelle operator \mathcal{L}_f . From the triangular inequality follows that

$$\|\mathcal{L}_f(h_f) - \lambda_f h_f\|_{L^1(\nu_f)} \leq \|\mathcal{L}_f(h_f) - \mathcal{L}_{f_n}(h_f)\|_{L^1(\nu_f)} + \|\mathcal{L}_{f_n}(h_f) - \lambda_f h_f\|_{L^1(\nu_f)}.$$

The Proposition 6 implies that the first term goes to zero when n goes to infinity. For the second term can estimate as follows

$$\begin{aligned} \|\mathcal{L}_{f_n}(h_f) - \lambda_f h_f\|_{L^1(\nu_f)} &\leq \|\mathcal{L}_{f_n}(h_f - h_{f_n} + h_{f_n}) - \lambda_f h_f\|_{L^1(\nu_f)} \\ &\leq \|\mathcal{L}_{f_n}(h_f - h_{f_n}) + \lambda_{f_n} h_{f_n} - \lambda_f h_f\|_{L^1(\nu_f)} \\ &\leq \|\mathcal{L}_{f_n}\|_{L^1(\nu)} \cdot \|h_f - h_{f_n}\|_{L^1(\nu)} + \|\lambda_{f_n} h_{f_n} - \lambda_f h_f\|_{L^1(\nu_f)}. \end{aligned}$$

Since $\sup_{n \in \mathbb{N}} \|\mathcal{L}_{f_n}\|_{L^1(\nu)} < +\infty$ and $\|h_f - h_{f_n}\|_{L^1(\Omega, \mathcal{F}, \nu_f)} \rightarrow 0$, when $n \rightarrow \infty$, we have that the first term in rhs also goes to zero when n goes to infinity. The second term in rhs above is bounded by

$$\begin{aligned} \|\lambda_{f_n} h_{f_n} - \lambda_f h_f\|_{L^1(\nu_f)} &\leq \|\lambda_{f_n} h_{f_n} - \lambda_f h_{f_n}\|_{L^1(\nu_f)} + \|\lambda_f h_{f_n} - \lambda_f h_f\|_{L^1(\nu_f)} \\ &= |\lambda_{f_n} - \lambda_f| \cdot \|h_{f_n}\|_{L^1(\nu_f)} + \|\lambda_f h_{f_n} - \lambda_f h_f\|_{L^1(\nu_f)}. \end{aligned}$$

From Corollary 2 and our assumption follows that the lhs above can be made small if n is big enough. Piecing together all these estimates we can conclude that $\|\mathcal{L}_f(h_f) - \lambda_f h_f\|_{L^1(\nu_f)} = 0$ and therefore $\mathcal{L}_f(h_f) = \lambda_f h_f$, ν_f a.s.. \square

Theorem 7. *Let μ_{f_n} as in (13) and suppose that $\mu_{f_n} \rightharpoonup \mu$, f_n Holder approximating f . If $\mu \ll \nu_f$ and $h_{f_n}(x) \rightarrow d\mu/d\nu_f$ ν_f -a.s. then $\mathcal{L}_f(d\mu/d\nu_f) = \lambda_f d\mu/d\nu_f$.*

Proof. Notice that

$$\int_{\Omega} |h_{f_n}| d\nu_f = 1 = \int_{\Omega} \left| \frac{d\mu}{d\nu_f} \right| d\nu_f \quad \text{and} \quad h_{f_n}(x) \rightarrow d\mu/d\nu_f \quad \nu_f - \text{a.s.}$$

The Scheffe's lemma implies that h_{f_n} converges to $d\mu/d\nu_f$ in the $L^1(\Omega, \mathcal{F}, \nu_f)$ norm. To finish the proof it is enough to apply the previous theorem. \square

We now build an eigenfunction for \mathcal{L}_f without assuming converge of h_{f_n} neither in $L^1(\Omega, \mathcal{F}, \nu_f)$ or almost surely sense. We should remark that the next theorem applies even when no convergent subsequence of $(h_{f_n})_{n \in \mathbb{N}}$ do exists in both senses.

Theorem 8. *Let $(h_{f_n})_{n \in \mathbb{N}}$ be a sequence of eigenfunctions on the unit sphere of $L^1(\Omega, \mathcal{F}, \nu_f)$, f_n Holder approximating f . If $\sup_{n \in \mathbb{N}} \|h_{f_n}\|_{\infty} < +\infty$, then $\limsup h_{f_n} \in L^1(\Omega, \mathcal{F}, \nu_f) \setminus \{0\}$ and moreover $\mathcal{L}_f(\limsup h_{f_n}) = \lambda_f \limsup h_{f_n}$.*

Proof. Since we are assuming that $\sup_{n \in \mathbb{N}} \|h_{f_n}\|_{\infty} < +\infty$ then $\limsup h_n \in L^1(\Omega, \mathcal{F}, \nu_f)$. For any fixed $x \in \Omega$ follows from this uniform bound that the mapping

$$M \ni a \mapsto \limsup_{n \rightarrow \infty} h_{f_n}(ax)$$

is uniformly bounded and therefore integrable with respect to the a-priori measure ν so can apply the limit sup version of the Fatou's lemma to get the following inequality

$$\begin{aligned} \mathcal{L}_f(\limsup_{n \rightarrow \infty} h_{f_n}) &= \int_M \exp(f(ax)) \limsup_{n \rightarrow \infty} h_{f_n}(ax) d\nu \\ &= \int_M \lim_{n \rightarrow \infty} \exp(f_n(ax)) \limsup_{n \rightarrow \infty} h_{f_n}(ax) d\nu \\ &= \int_M \limsup_{n \rightarrow \infty} (\exp(f_n(ax)) h_{f_n}(ax)) d\nu \\ &\geq \limsup_{n \rightarrow \infty} \int_M \exp(f_n(ax)) h_{f_n}(ax) d\nu \\ &= \limsup_{n \rightarrow \infty} \lambda_{f_n} h_{f_n} \\ &= \lambda_f \limsup_{n \rightarrow \infty} h_{f_n}. \end{aligned}$$

These inequalities implies that $\limsup h_{f_n}$ is a super solution to the eigenvalue problem. On the other hand, we have proved that the operator norm $\|\mathcal{L}_f\|_{L^1(\nu_f)} = \lambda_f$. This fact together with the previous inequality implies, ν_f almost surely, that

$$\mathcal{L}_f(\limsup_{n \rightarrow \infty} h_{f_n}) = \lambda_f \limsup_{n \rightarrow \infty} h_{f_n}.$$

Remains to argue that $\limsup h_{f_n}$ is non trivial. Since we are assuming that $\sup_{n \in \mathbb{N}} \|h_{f_n}\|_{\infty} < +\infty$ we can ensure that $\mu_{f_n} \rightharpoonup \mu \ll \nu_f$. Indeed, for any open set $A \subset \Omega$ follows from the weak convergence and the Portmanteau Theorem that

$$\mu(A) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} 1_A h_{f_n} d\nu_f.$$

Since ν_f is outer regular we have for any $B \in \mathcal{F}$ that $\nu_f(B) = \inf\{\nu_f(A) : A \supset B, A \text{ open}\}$. From the previous inequality and uniform limitation of h_{f_n} we get for any $B \subset A$ (A open set) that $\mu(B) \leq \mu(A) \leq \sup_{n \in \mathbb{N}} \|h_{f_n}\|_\infty \nu_f(B)$ and thus $\mu \ll \nu_f$. By applying again the limit sup version of the Fatou Lemma we get that

$$1 = \int_{\Omega} \frac{d\mu}{d\nu_f} d\nu_f = \lim_{n \rightarrow \infty} \int_{\Omega} h_{f_n} d\nu_f = \limsup_{n \rightarrow \infty} \int_{\Omega} h_{f_n} d\nu_f \leq \int_{\Omega} \limsup_{n \rightarrow \infty} h_{f_n} d\nu_f,$$

where the second equality comes from the definition of the weak convergence. \square

We point out that the condition: $\sup_{n \in \mathbb{N}} \|h_{f_n}\|_\infty < +\infty$, f_n Holder approximating f , is not true for Hofbauer potentials.

All the previous theorems of this section requires information about the eigenfunction. Now we present existence result that one can check by using only the potential (via \mathcal{L}_f^n) and some estimates on the maximal eigenvalue.

Theorem 9. *Let f be a continuous potential and λ_f the eigenvalue of \mathcal{L}_f^* provided by Proposition 1. If*

$$\sup_{n \in \mathbb{N}} \|\mathcal{L}_f^n(1)/\lambda_f^n\|_\infty < +\infty,$$

then, $\limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)/\lambda_f^n$ is a non trivial $L^1(\Omega, \mathcal{F}, \nu_f)$ eigenfunction of \mathcal{L}_f associated to λ_f .

Proof. The key idea is to prove that $\limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)/\lambda_f^n$ is a super solution for the eigenvalue problem, since it belongs to $L^1(\Omega, \mathcal{F}, \nu_f)$ it has to be a sub solution and then it is in fact a solution. Its non-triviality is based on the arguments given in the previous proof and the weak convergence of suitable sequence of probability measures.

The super solution part of the argument is again based on the reverse Fatou Lemma as follows

$$\begin{aligned} \mathcal{L}_f(\limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)/\lambda_f^n) &= \int_M \exp(f(ax)) \limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)(ax)/\lambda_f^n d\nu \\ &\geq \limsup_{n \rightarrow \infty} \int_M \exp(f(ax)) \mathcal{L}_f^n(1)(ax)/\lambda_f^n d\nu \\ &= \limsup_{n \rightarrow \infty} \lambda_f \mathcal{L}_f^{n+1}(1)(x)/\lambda_f^{n+1} \\ &= \lambda_f \limsup_{n \rightarrow \infty} \mathcal{L}_f^n(1)(x)/\lambda_f^n. \end{aligned}$$

The next step is to prove the non-triviality of this limsup. From the definition of ν_f we can say that the following sequence is contained in $\mathcal{P}(\Omega, \mathcal{F})$

$$\mathcal{F} \ni E \mapsto \int_E \frac{\mathcal{L}_f^n(1)}{\lambda_f^n} d\nu_f.$$

Similarly, to the previous theorem we can ensure that all its cluster points in the weak topology are absolutely continuous with respect to ν_f . Up to subsequence, we can see

by another application of the Fatou Lemma that

$$1 = \int_{\Omega} \frac{d\mu}{d\nu_f} d\nu_f = \lim_{n \rightarrow \infty} \int_{\Omega} \frac{\mathcal{L}_f^n(1)}{\lambda_f^n} d\nu_f \leq \int_{\Omega} \limsup_{n \rightarrow \infty} \frac{\mathcal{L}_f^n(1)}{\lambda_f^n} d\nu_f.$$

□

Assume that the state space $M = \{-1, 1\}$ and the a priori measure is the uniform probability measure. Consider the continuous potential f given by $f(x) = \sum_{k \geq 1} (x_k/k^\gamma)$, where $\gamma \geq 2$. For such choices of the exponent γ this potential f is continuous but not Hölder. If $f_n(x) = \sum_{k=1}^n (x_k/k^\gamma)$, $n \in \mathbb{N}$, then the hypothesis of the last theorem are in hold.

11 Applications

Weak Convergence of Eigenprobabilities

In this section we consider $f : \Omega \rightarrow \mathbb{R}$ be a continuous potential or an element of $C^\gamma(\Omega)$ for some $0 \leq \gamma < 1$. We would like to get results for continuous potentials via limits of Hölder potentials.

We choose a point in the state space M and for simplicity call it 0. We denote by $(f_n)_{n \in \mathbb{N}} \subset C^\gamma(\Omega)$ the sequence given by $f_n(x) = f(x_1, \dots, x_n, 0, 0, \dots)$. Keeping notation of the previous sections eigenprobabilities of \mathcal{L}_{f_n} and \mathcal{L}_f are also denoted by ν_{f_n} and ν_f , respectively. Notice that $\|f - f_n\|_\infty \rightarrow 0$, when $n \rightarrow \infty$ and moreover if f is Hölder then this convergence is exponentially fast. We denote by $\mathcal{L}(C(\Omega))$ the space of all bounded operators from $C(\Omega)$ to itself and for each $T \in \mathcal{L}(C(\Omega))$ we use the notation $\|T\|_{C(\Omega)}$ for its operator norm. The next lemma is inspired in the Proposition 6.

Lemma 4. *The sequence $(\mathcal{L}_{f_n})_{n \in \mathbb{N}}$ converges in the operator norm to the Ruelle operator \mathcal{L}_f , i.e., $\|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{C(\Omega)} \rightarrow 0$, when $n \rightarrow \infty$.*

Proof. For all $n \in \mathbb{N}$ we have

$$\begin{aligned} \|\mathcal{L}_{f_n} - \mathcal{L}_f\|_{C(\Omega)} &= \sup_{0 < \|\varphi\|_\infty \leq 1} \sup_{x \in \Omega} |\mathcal{L}_{f_n}(\varphi)(x) - \mathcal{L}_f(\varphi)(x)| \\ &\leq \sup_{0 < \|\varphi\|_\infty \leq 1} \sup_{x \in \Omega} |\mathcal{L}_f(\exp(f_n - f)\varphi) - \mathcal{L}_f(\varphi)| \\ &\leq \|\mathcal{L}_f\|_{C(\Omega)} \sup_{0 < \|\varphi\|_\infty \leq 1} \|\varphi\|_\infty \|\exp(f_n - f) - 1\|_\infty \\ &\leq \|\mathcal{L}_f\|_{C(\Omega)} \|\exp(f_n - f) - 1\|_\infty. \end{aligned}$$

□

Proposition 7. *Any cluster point, in the weak topology, of the sequence $(\nu_{f_n})_{n \in \mathbb{N}}$ belongs to the set $\mathcal{G}^*(f)$.*

Proof. By the previous lemma for any given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ we have for all $\varphi \in C(\Omega)$ and for all $x \in \Omega$ that $\mathcal{L}_{f_n}(\varphi)(x) - \varepsilon < \mathcal{L}_f(\varphi)(x) < \mathcal{L}_{f_n}(\varphi)(x) + \varepsilon$. From the duality relation of the Ruelle operator and the weak convergence and Corollary 2 we have that

$$\begin{aligned} \int_{\Omega} \varphi d(\mathcal{L}_f^* \nu) &= \int_{\Omega} \mathcal{L}_f(\varphi) d\nu = \lim_{n \rightarrow \infty} \int_{\Omega} \mathcal{L}_f(\varphi) d\nu_{f_n} \\ &< \lim_{n \rightarrow \infty} \int_{\Omega} \mathcal{L}_{f_n}(\varphi) d\nu_{f_n} + \varepsilon \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} \varphi d(\mathcal{L}_{f_n} \nu_{f_n}) + \varepsilon \\ &= \lim_{n \rightarrow \infty} \lambda_{f_n} \int_{\Omega} \varphi d\nu_{f_n} + \varepsilon \\ &= \lambda_f \int_{\Omega} \varphi d\nu + \varepsilon. \end{aligned}$$

We obtain analogous lower bound, with $-\varepsilon$ instead. Since $\varepsilon > 0$ is arbitrary follows for any $\varphi \in C(\Omega)$ that

$$\int_{\Omega} \varphi d(\mathcal{L}_f^* \nu) = \lambda_f \int_{\Omega} \varphi d\nu$$

and therefore $\mathcal{L}_f^* \nu = \lambda_f \nu$. □

Remark 3. *The above proposition for $f \in C^{\gamma}(\Omega)$ says that up to subsequence $\nu_{f_n} \rightharpoonup \nu_f$, which is the unique eigenprobability of \mathcal{L}_f^* . Therefore the eigenprobability ν_f inherits all the properties of the sequence ν_{f_n} that are preserved by weak limits.*

Constructive Approach for Equilibrium States for General Continuous Potentials

Lemma 5. *For each $n \in \mathbb{N}$ let f_n be the potential above defined, h_{f_n} denotes the main eigenfunction of \mathcal{L}_{f_n} associated to λ_{f_n} normalized so that $\|h_{f_n}\|_{L^1(\nu_{f_n})} = 1$, where ν_{f_n} is the unique eigenprobability of $\mathcal{L}_{f_n}^*$. Then there exist a σ -invariant Borel probability measure μ_f such that, up to subsequence,*

$$\lim_{n \rightarrow \infty} \int_{\Omega} \varphi h_n d\nu_n = \int_{\Omega} \varphi d\mu_f, \quad \forall \varphi \in C(\Omega)$$

Proof. It is well known that $h_{f_n} \nu_{f_n}$ defines a σ -invariant Borel probability measure and therefore any of its cluster point, in the weak topology is a shift invariant probability measure. □

Theorem 10 (Equilibrium States). *Let $f : \Omega \rightarrow \mathbb{R}$ be a continuous potential and $(f_n)_{n \in \mathbb{N}}$ a sequence of Hölder potentials such that $\|f_n - f\|_{\infty} \rightarrow 0$, when $n \rightarrow \infty$. Then any probability measure μ_f as constructed in the Lemma 5 is an equilibrium state for f .*

Proof. Given any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ then $-\varepsilon < f_n - f < \varepsilon$. We know that equilibrium measure μ_{f_n} for f_n is given by $\mu_{f_n} = h_{f_n} \nu_{f_n}$ and therefore we have the following inequality

$$\begin{aligned} \sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) - \int_\Omega f d\rho \right\} &= \sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) + \int_\Omega (f_n - f) d\rho - \int_\Omega f_n d\rho \right\} \\ &< \varepsilon + \sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) - \int_\Omega f_n d\rho \right\} \\ &= \varepsilon + h(\mu_{f_n}) - \int_\Omega f_n d\mu_{f_n}. \end{aligned}$$

Since the Kolmogorov-Sinai entropy is upper semi-continuous and $\mu_{f_n} \rightharpoonup \mu_f$ it follows that for some $n_1 \in \mathbb{N}$ and $n \geq n_1$ we have

$$h(\mu_{f_n}) < h(\mu_f) + \varepsilon.$$

Using again the uniform convergence of f_n to f and the weak convergence of μ_{f_n} to μ_f , for some $n_2 \in \mathbb{N}$ and $n \geq n_2$ we get

$$- \int_\Omega f_n d\mu_{f_n} = \int_\Omega f - f_n d\mu_{f_n} - \int_\Omega f d\mu_{f_n} < \varepsilon - \int_\Omega f d\mu_{f_n} < 2\varepsilon - \int_\Omega f d\mu_{f_n}.$$

Using the previous three inequalities we get for $n \geq \max\{n_0, n_1, n_2\}$

$$\sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) - \int_\Omega f d\rho \right\} < 4\varepsilon + h(\mu_f) - \int_\Omega f d\mu_f.$$

Since $\varepsilon > 0$ is arbitrary follows from the definition of the supremum and above inequality that

$$\sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) - \int_\Omega f d\rho \right\} = h(\mu_f) - \int_\Omega f d\mu_f$$

and therefore μ_f constructed in Lemma 5 is an equilibrium state. \square

Corollary 3. *For any continuous potential $f : \Omega \rightarrow \mathbb{R}$ we have that*

$$\log \lambda_f = \sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) - \int_\Omega f d\rho \right\}.$$

Proof. Consider the Hölder approximations $(f_n)_{n \in \mathbb{N}}$ of f as above. Then for any given $\varepsilon > 0$ and n large enough we have

$$\begin{aligned} \log \lambda_{f_n} - \varepsilon &= h(\mu_{f_n}) - \int_\Omega f_n d\mu_{f_n} - \varepsilon \\ &< \sup_{\rho \in \mathcal{P}_\sigma(\Omega, \mathcal{F})} \left\{ h(\rho) - \int_\Omega f d\rho \right\} \end{aligned}$$

$$\begin{aligned}
&< \varepsilon + h(\mu_{f_n}) - \int_{\Omega} f_n d\mu_{f_n} \\
&= \varepsilon + \log \lambda_{f_n}.
\end{aligned}$$

Since $\lambda_{f_n} \rightarrow \lambda_f$ it follows from the above inequality that

$$\sup_{\rho \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})} \left\{ h(\rho) - \int_{\Omega} f d\rho \right\} = \log \lambda_f.$$

□

Necessary and Sufficient Conditions for the Existence of L^1 Eigenfunctions

Theorem 11. *Let $\nu \in \mathcal{G}^*(f)$. The Ruelle operator has a non-negative eigenfunction $h \in L^1(\nu)$ if, and only if, there exists $\mu \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})$ such that $\mu \ll \nu$.*

Proof. We first assume that there is $\mu \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})$ so that $\mu \ll \nu$. In this case we claim that

$$\mathcal{L}_f \left(\frac{d\mu}{d\nu} \right) = \lambda_f \frac{d\mu}{d\nu}.$$

Indeed, for any continuous function φ we have

$$\begin{aligned}
\int_{\Omega} \varphi \mathcal{L}_f \left(\frac{d\mu}{d\nu} \right) d\nu &= \int_{\Omega} \mathcal{L}_f \left(\varphi \circ \sigma \cdot \frac{d\mu}{d\nu} \right) d\nu \\
&= \lambda_f \int_{\Omega} \varphi \circ \sigma \cdot \frac{d\mu}{d\nu} d\nu = \lambda_f \int_{\Omega} \varphi \circ \sigma \cdot d\mu \\
&= \lambda_f \int_{\Omega} \varphi d\mu = \lambda_f \int_{\Omega} \varphi \cdot \frac{d\mu}{d\nu} d\nu.
\end{aligned}$$

Conversely, suppose that $h \in L^1(\nu)$ is a non-negative eigenfunction for the Ruelle operator associated to the main eigenvalue and normalized so that $\int_{\Omega} h d\nu = 1$. Define the probability measure $\mu = h\nu$. Then for any $\varphi \in C(\Omega)$ we have

$$\begin{aligned}
\lambda_f \int_{\Omega} \varphi d\mu &= \lambda_f \int_{\Omega} \varphi h d\nu = \int_{\Omega} \varphi \mathcal{L}_f h d\nu \\
&= \int_{\Omega} \mathcal{L}_f(\varphi \circ \sigma \cdot h) d\nu = \lambda_f \int_{\Omega} \varphi \circ \sigma \cdot h d\nu \\
&= \lambda_f \int_{\Omega} \varphi \circ \sigma d\mu
\end{aligned}$$

and therefore $\mu \in \mathcal{P}_{\sigma}(\Omega, \mathcal{F})$ and $\mu \ll \nu$.

□

Continuous Potentials Having Only L^1 Eigenfunctions

Following the results of [7] now we assume that the state space $M = \{-1, 1\}$ and the a priori measure is the uniform probability measure, which we denote by κ . Let ρ be the infinite product measure $\rho = \prod_{i \in \mathbb{N}} \kappa$. Consider the continuous potential f given by $f(x) = \sum_{n \geq 1} (x_n/n^\gamma)$, where $3/2 < \gamma \leq 2$. For each $n \in \mathbb{N}$ set $\alpha_n = \zeta(\gamma) - \sum_{j=1}^n n^{-\gamma}$. The main eigenvalue for \mathcal{L}_f is $\lambda_f = 2 \cosh(\zeta(\gamma))$ and there is a \mathcal{F} -measurable set $\Omega_0 \subset \Omega$ satisfying $\rho(\Omega_0) = 1$ and such that for all $x \in \Omega_0$ the following function

$$x \mapsto h_f(x) \equiv \exp(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \dots + \alpha_n x_n + \dots)$$

is well defined. One can show that h_f is the unique eigenfunction associated to λ_f and it is not an element of $L^\infty(\Omega, \mathcal{F}, \rho)$ which implies that $h_f \notin C(\Omega)$. We can also prove that $h_f \in L^1(\Omega, \mathcal{F}, \nu_f)$.

12 Appendix

On this appendix we adapt some results from the reference [12] to the present setting. Let \mathcal{L}_f be the Ruelle operator of a continuous potential f and for each $n \in \mathbb{N}$, $x \in \Omega$ and $E \in \mathcal{F}$, consider the probability kernel $K_n : \mathcal{F} \times \Omega \rightarrow [0, 1]$ given by the expression

$$K_n(E, x) \equiv \frac{\mathcal{L}_f^n(1_E)(\sigma^n(x))}{\mathcal{L}_f^n(1)(\sigma^n(x))}.$$

Proposition 8. *Suppose $\mu \in \mathcal{G}^{DLR}(f)$, then for all $n \in \mathbb{N}$*

$$\mathcal{A}_n(\mu) = \{E \in \mathcal{F} : K_n(E, \omega) = 1_E(\omega) \text{ } \mu \text{ a.s.}\} \text{ is a } \sigma\text{-algebra.}$$

Proof. Since $K_n(\Omega, \omega) = 1 = 1_\Omega(\omega)$ we get that $\Omega \in \mathcal{A}_n(\mu)$. For the empty set the proof is trivial.

Suppose $(E_j)_{j \in \mathbb{N}}$ is a disjoint collection of elements of $\mathcal{A}_n(\mu)$. Then, for all ω we get $K_n(\cup_{j \in \mathbb{N}} E_j, \omega) = \sum_{j \in \mathbb{N}} K_n(E_j, \omega)$. Note that μ -a.e. $K_n(E_j, \omega) = 1_{E_j}(\omega)$ for all $j \in \mathbb{N}$, because $E_j \in \mathcal{A}_n(\mu)$. Clearly, $1_{\cup_{j \in \mathbb{N}} E_j}(\omega) = \sum_{j \in \mathbb{N}} 1_{E_j}(\omega)$, then by using that the intersection of sets of measure one has measure one, we get that $K_n(\cup_{j \in \mathbb{N}} E_j, \omega) = 1_{\cup_{j \in \mathbb{N}} E_j}(\omega)$, μ -a.e..

Note that $\mathcal{A}_n(\mu)$ is closed by the complement operation. Indeed, for all $\omega \in \Omega$ and $E \in \mathcal{A}_n(\mu)$ we have that $K_n(E^c, \omega) = 1 - K_n(E, \omega) = 1 - 1_E(\omega) = 1_{E^c}(\omega)$.

Since we have shown that $\mathcal{A}_n(\mu)$ is closed under denumerable disjoint unions then the remaining task is to show that $\mathcal{A}_n(\mu)$ is closed under finite intersections. Then it will follow that $\mathcal{A}_n(\mu)$ is closed under any denumerable union. Suppose that $E, F \in \mathcal{A}_n(\mu)$. By the monotonicity of the measure we have μ -a.e that

$$\begin{aligned} K_n(E \cap F, \omega) &\leq \min\{K_n(E, \omega), K_n(F, \omega)\} \\ &= \min\{1_E(\omega), 1_F(\omega)\} \\ &= 1_{E \cap F}(\omega). \end{aligned}$$

By using the hypothesis we get that

$$\begin{aligned}
\int_{\Omega} [1_{E \cap F} - K_n(E \cap F, \cdot)] d\mu &= \int_{\Omega} 1_{E \cap F} d\mu - \int_{\Omega} K_n(E \cap F, \cdot) d\mu \\
&= \mu(E \cap F) - \int_{\Omega} K_n(E \cap F, \cdot) d\mu \\
&= \mu(E \cap F) - \mu(E \cap F) \\
&= 0.
\end{aligned}$$

From the previous inequality we know that the integrand in the left hand side of the above is non-negative. So it has to be zero μ -a.e.. Therefore, $K_n(E \cap F, \omega) = 1_{E \cap F}(\omega)$, μ -a.e.. and finally we get that $\mathcal{A}_n(\mu)$ is closed for finite intersections. Therefore, $\mathcal{A}_n(\mu)$ is a σ -algebra. \square

Proposition 9. *Given a function $g : \Omega \rightarrow [0, \infty)$ we get the equivalence:*

- 1- $\int_{\Omega} K_n(E, \cdot) g d\mu = \int_{\Omega} 1_E g d\mu$ for all $E \in \mathcal{F}$,
- 2- The function g is measurable with respect to the sigma-algebra $\mathcal{A}_n(\mu)$.

Remark 4. In [12] the condition 1 is denoted by $(g\mu)K_n = g\mu$, where $g\mu$ is the measure defined by $E \mapsto \int_{\Omega} 1_E g d\mu$. This condition is equivalent to say that $g\mu$ is compatible with K_n .

Proof. First we will prove that $1 \implies 2$. This follows from the following claim: for all $g : \Omega \rightarrow [0, \infty)$ for which the condition 1 holds, we have $\{g \geq c\} \in \mathcal{A}_n(\mu)$, for any $c \in \mathbb{R}$. Indeed, the identity $1_{\{g < c\}} = 1 - 1_{\{g \geq c\}}$ implies

$$\begin{aligned}
&\int_{\{g < c\}} K_n(1_{\{g \geq c\}}, \omega) g(\omega) d\mu(\omega) \\
&= \int_{\Omega} K_n(1_{\{g \geq c\}}, \omega) g(\omega) d\mu(\omega) - \int_{\Omega} 1_{\{g \geq c\}}(\omega) g(\omega) K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega).
\end{aligned}$$

By using the condition 1 in the first expression of rhs we get

$$\begin{aligned}
&\int_{\{g < c\}} K_n(1_{\{g \geq c\}}, \omega) g(\omega) d\mu(\omega) \\
&= \int_{\Omega} 1_{\{g \geq c\}}(\omega) g(\omega) d\mu(\omega) - \int_{\Omega} 1_{\{g \geq c\}}(\omega) g(\omega) K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) \\
&= \int_{\Omega} 1_{\{g \geq c\}}(\omega) g(\omega) (1 - K_n(1_{\{g \geq c\}}, \omega)) d\mu(\omega).
\end{aligned}$$

Now, we will use the two inequalities $1_{\{g \geq c\}}(\omega) g(\omega) \geq c \cdot 1_{\{g \geq c\}}(\omega)$ and $K_n(1_{\{g \geq c\}}, \omega) \leq 1$, in the above expression, to get

$$\int_{\{g < c\}} K_n(1_{\{g \geq c\}}, \omega) g(\omega) d\mu(\omega)$$

$$\begin{aligned}
&= \int_{\Omega} 1_{\{g \geq c\}}(\omega) g(\omega) (1 - K_n(1_{\{g \geq c\}}, \omega)) d\mu(\omega) \\
&\geq c \int_{\Omega} 1_{\{g \geq c\}}(\omega) (1 - K_n(1_{\{g \geq c\}}, \omega)) d\mu(\omega) \\
&= c \int_{\Omega} 1_{\{g \geq c\}}(\omega) d\mu(\omega) - c \int_{\Omega} 1_{\{g \geq c\}}(\omega) K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) \\
&\stackrel{(cond 1)}{=} c \int_{\Omega} K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) - c \int_{\Omega} 1_{\{g \geq c\}}(\omega) K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) \\
&\stackrel{(1_{\{g < c\}} = 1 - 1_{\{g \leq c\}})}{=} c \int_{\{g < c\}} K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega).
\end{aligned}$$

Now, the two extremes of the above inequality give us

$$\int_{\{g < c\}} (g - c) K_n(1_{\{g \geq c\}}, \omega) d\mu(\omega) \geq 0.$$

Therefore, $1_{\{g < c\}}(\omega) K_n(1_{\{g \geq c\}}, \omega) = 0$ μ -a.e.. From this follows that

$$\begin{aligned}
K_n(1_{\{g \geq c\}}, \omega) &= 1_{\{g \geq c\}}(\omega) K_n(1_{\{g \geq c\}}, \omega) + 1_{\{g < c\}}(\omega) K_n(1_{\{g \geq c\}}, \omega) \\
&= 1_{\{g \geq c\}}(\omega) K_n(1_{\{g \geq c\}}, \omega) \\
&\leq 1_{\{g \geq c\}}(\omega).
\end{aligned}$$

By another application of the condition 1 we get

$$\int_{\Omega} 1_{\{g \geq c\}}(\omega) - K_n(1_{\{g \geq c\}}, \omega) d\mu = 0$$

and, then from the last inequality we obtain the μ -a.e. equality $1_{\{g \geq c\}}(\omega) = K_n(1_{\{g \geq c\}}, \omega)$. This means that $\{g \geq c\} \in \mathcal{A}_n(\mu)$ and so g is $\mathcal{A}_n(\mu)$ -measurable.

Now we will show that $2 \implies 1$. Suppose g is $\mathcal{A}_n(\mu)$ -measurable. First we will show that $2 \implies 1$ holds when $g = 1_F$, for some $F \in \mathcal{A}_n(\mu)$. To prove this claim, it only remains to verify that

$$\int_{\Omega} 1_F \cdot K_n(E, \cdot) d\mu = \int_{\Omega} 1_F \cdot 1_E d\mu, \quad \forall E \in \mathcal{F}. \quad (15)$$

Note that for any $E \in \mathcal{F}$ we have

$$\begin{aligned}
\int_{\Omega} 1_F \cdot K_n(E, \cdot) d\mu &= \int_{\Omega} 1_F \cdot K_n(E \cap F, \cdot) d\mu + \int_{\Omega} 1_F \cdot K_n(E \cap F^c, \cdot) d\mu \\
&\leq \int_{\Omega} K_n(E \cap F, \cdot) d\mu + \int_{\Omega} 1_F \cdot K_n(F^c, \cdot) d\mu \\
&\stackrel{(\text{Hip. on } K_n)}{=} \int_{\Omega} 1_{E \cap F} d\mu + \int_{\Omega} 1_F \cdot K_n(F^c, \cdot) d\mu
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(F \in \mathcal{A}_n(\mu))}{=} \int_{\Omega} 1_{E \cap F} d\mu + \int_{\Omega} 1_F \cdot 1_{F^c} d\mu \\
&= \int_{\Omega} 1_E \cdot 1_F d\mu.
\end{aligned}$$

By a similar argument we can show that

$$\int_{\Omega} 1_F \cdot K_n(E^c, \cdot) d\mu \leq \int_{\Omega} 1_{E^c} \cdot 1_F d\mu.$$

Since

$$\begin{aligned}
\int_{\Omega} 1_F \cdot K_n(E, \cdot) d\mu + \int_{\Omega} 1_F \cdot K_n(E^c, \cdot) d\mu &= \mu(F) \\
&= \int_{\Omega} 1_F \cdot 1_E d\mu + \int_{\Omega} 1_F \cdot 1_{E^c} d\mu
\end{aligned}$$

it follows from the two last inequalities that

$$\int_{\Omega} 1_F \cdot K_n(E, \cdot) d\mu = \int_{\Omega} 1_F \cdot 1_E d\mu, \quad \forall E \in \mathcal{F}.$$

The above identity extends by linearity for simple functions. By taking a sequence of simple functions $\varphi_k \uparrow f$, and using the monotone convergence theorem we get for any $\mathcal{A}_n(\mu)$ -measurable function g that

$$\int_{\Omega} g \cdot K_n(E, \cdot) d\mu = \int_{\Omega} g \cdot 1_E d\mu, \quad \forall E \in \mathcal{F}.$$

□

It follows from last proposition that if g is measurable with respect to the sigma-algebra $\mathcal{A}_n(\mu)$ for all $n \in \mathbb{N}$, and $\mu \in \mathcal{G}^{DLR}(f)$, f continuous, then $g\mu$ is also in $\mathcal{G}^{DLR}(f)$.

Corollary 4. *Given $\mu \in \mathcal{G}^{DLR}(f)$ define $\mathcal{A}(\mu) \equiv \bigcap_{n \in \mathbb{N}} \mathcal{A}_n(\mu)$. Then, μ is extreme in $\mathcal{G}^{DLR}(f)$, if and only if, μ is trivial on $\mathcal{A}(\mu)$.*

Proof. Suppose that there exists $F \in \mathcal{A}(\mu)$ such that $0 < \mu(F) < 1$ and consider the following probability measures

$$\begin{aligned}
\mathcal{F} \ni E &\mapsto \nu(E) = \mu(E|F) = \int_{\Omega} \frac{1}{\mu(F)} 1_E 1_F d\mu, \\
\mathcal{F} \ni E &\mapsto \gamma(E) = \mu(E|F^c) = \int_{\Omega} \frac{1}{\mu(F^c)} 1_E 1_{F^c} d\mu.
\end{aligned}$$

Clearly $\nu \neq \gamma$ and moreover

$$\mu = \mu(F)\nu + (1 - \mu(F))\gamma. \tag{16}$$

The last proposition guarantees that both ν and γ belong to $\mathcal{G}^{DLR}(f)$. Indeed, in last proposition take f as $(1/\mu(F)) \cdot 1_F$ and $(1/\mu(F^c)) \cdot 1_{F^c}$, respectively (these functions are $\mathcal{A}_n(\mu)$ -measurable for $n \in \mathbb{N}$). However the existence of the non trivial convex combination (16), of two elements in $\mathcal{G}^{DLR}(f)$, is a contradiction. Therefore, any set $F \in \mathcal{A}(\mu)$ has the μ measure zero or one.

Conversely, suppose that μ is trivial on $\mathcal{A}(\mu)$ and at same time expressible as $\mu = \lambda\nu + (1 - \lambda)\gamma$, with $0 < \lambda < 1$ and $\nu, \gamma \in \mathcal{G}(f)$.

Note that $\nu \ll \mu$ and then from Radon-Nikodym Theorem we get that $\nu(E) = \int_{\Omega} 1_E f d\mu$ for some measurable function $f \geq 0$. Once more by the equivalence $1 \iff 2$ we get that f is $\mathcal{A}_n(\mu)$ -measurable for all $n \in \mathbb{N}$ (recall that $\nu \in \mathcal{G}^{DLR}(f)$). Since we assumed that μ is trivial on $\mathcal{A}(\mu)$ we get that both integrals below are always equals to each other being zero or one

$$\int_{\Omega} 1_F f d\mu = \int_{\Omega} 1_F d\mu, \quad \forall F \in \mathcal{A}(\mu).$$

As the equality is valid for all $F \in \mathcal{A}(\mu)$ and f is $\mathcal{A}_n(\mu)$ -measurable we can conclude that $f = 1$ μ -a.e.. Therefore, $\mu = \nu$ and consequently $\gamma = \mu$. So μ is a extreme point of $\mathcal{G}^{DLR}(f)$. \square

It follows from last corollary that if $\mathcal{G}^{DLR}(f)$ has only one element μ , then μ is trivial on $\mathcal{A}(\mu)$. If there is phase transition, in the sense that the cardinality of $\mathcal{G}^{DLR}(f)$ is bigger than one, then any extreme probability measure μ in $\mathcal{G}^{DLR}(f)$ is trivial on $\mathcal{A}(\mu)$.

In the next proposition we show the relationship between $\mathcal{A}(\mu)$ and $\cap_{j \in \mathbb{N}} \sigma^j(\mathcal{F})$, for $\mu \in \mathcal{G}^{DLR}$.

Corollary 5. *If $\mu \in \mathcal{G}^{DLR}(f)$ then $\mathcal{A}(\mu)$ is a μ completion of $\cap_{j \in \mathbb{N}} \sigma^j(\mathcal{F})$. In particular, it follows from last corollary that if $\mu \in \mathcal{G}^{DLR}(f)$ is extreme, then, it is trivial on $\cap_{j \in \mathbb{N}} \sigma^j(\mathcal{F})$.*

Proof. For all $n \in \mathbb{N}$ we have that K_n is a proper kernel. Therefore, for any set $F \in \cap_{j \in \mathbb{N}} \sigma^j(\mathcal{F})$ we get that $K_n(F, \omega) = 1_F(\omega)$. On the other hand, if $F \in \{E \in \mathcal{F} : K_n(E, \omega) = 1_E(\omega), \forall n \in \mathbb{N}, \forall \omega \in \Omega\}$, then, $F = \{\omega \in \Omega : K_n(F, \omega) = 1\} \in \sigma^n(\mathcal{F})$. Therefore, $F \in \cap_{j \in \mathbb{N}} \sigma^j(\mathcal{F})$. Consider $\mu \in \mathcal{G}^{DLR}(f)$ and let $F \in \mathcal{A}(\mu)$, then,

$$B = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \{\omega \in \Omega : K_m(F, \omega) = 1\}$$

is an element on the sigma algebra $\cap_{j \in \mathbb{N}} \sigma^j(\mathcal{F})$ and moreover, $\mu(F \Delta B) = 0$, because

$$1_B = \limsup_{n \rightarrow \infty} 1_{\{\omega \in \Omega : K_n(F, \omega) = 1\}} = \limsup_{n \rightarrow \infty} 1_F = 1_F \quad \mu \text{ a.e..}$$

\square

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